Solutions to Exercises for the book "Infinite-dimensional Spectral Computations"

Abstract

This is a complete set of solutions to the exercises in the book, "INFINITE-DIMENSIONAL SPECTRAL COMPUTATIONS: Foundations, Algorithms, and Modern Applications". The solutions were provided by Matthew Colbrook, Gustav Conradie, George Coote, and April Herwig. Any corrections, suggestions or comments should be emailed to Matthew Colbrook at m.colbrook@damtp.cam.ac.uk. Some of the solutions are linked to research papers, in which case we have provided a reference and discussion.

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1 Chapter 1

Exercise 1.1

The sequence $\{\mathcal{P}_n^* v_n\}$ has a weakly convergent subsequence (by the Banach–Alaoglu theorem[1]). Without loss of generality, it is enough to assume that $\mathcal{P}_n^* v_n$ converges weakly to v and show that v must equal 0. Since A is bounded and $\mathcal{P}_n^*\mathcal{P}_n$ converges strongly to the identity, $\mathcal{P}_n^*\mathcal{P}_n A\mathcal{P}_n^* v_n$ converges weakly to Av. But $\mathcal{P}_n^*\mathcal{P}_n A\mathcal{P}_n^* v_n = \lambda_n \mathcal{P}_n^* v_n$, which converges weakly to λv. Hence, $Av = \lambda v$. Since $\lambda \notin \mathrm{Sp}(A)$, λ cannot be an eigenvalue and we must have $v = 0$.

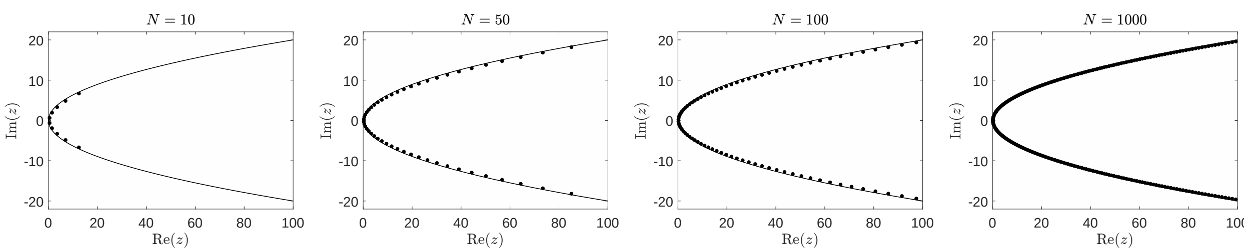
Exercise 1.2

Consider the Fourier transform defined by

$$[\mathcal{F} f](\omega) = \int_{\mathbb{R}} f(x)\exp(-2\pi \mathrm{i}\omega x)\,\mathrm{d}x,$$

for smooth compactly supported functions on $L^2(\mathbb{R})$. With this convention, $\mathcal{F}$ extends to a unitary transformation on the whole of $L^2(\mathbb{R})$. Let M be the (unbounded) multiplication operator $g(\omega) \mapsto ((2\pi\omega)^2 - 2\mathrm{i}(2\pi\omega))g(\omega)$ on $L^2(\mathbb{R})$. Then, $\mathcal{L} = \mathcal{F}^{-1}M\mathcal{F}$ and, hence, $\mathrm{Sp}(\mathcal{L}) = \mathrm{Sp}(M)$. If $z \notin \{k^2 + 2\mathrm{i}k : k \in \mathbb{R}\}$, then $((2\pi\omega)^2 - 2\mathrm{i}(2\pi\omega) - z)^{-1}$ is a bounded rational function of $\omega \in \mathbb{R}$, and hence, $(M - zI)^{-1}g(\omega) = ((2\pi\omega)^2 - 2\mathrm{i}(2\pi\omega) - z)^{-1}g(\omega)$ exists as a bounded operator. Similarly, we can prove that every $k^2 + 2\mathrm{i}k$ with $k \in \mathbb{R}$ lies in $\mathrm{Sp}(M)$. The first part of the exercise follows.

Using the recurrence relations for Hermite functions, we can represent $\mathcal{L}$ as an infinite banded matrix (see MATLAB code "ex1_2.m" in the repository). Here are the eigenvalues of finite sections to $N \times N$ matrices for various N, along with $\mathrm{Sp}(\mathcal{L})$ shown as a solid line:



The finite section method converges for this example.

Exercise 1.3

For the first part, we can identify a sequence $\{x_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ with the Fourier series of a function $f \in L^2(\mathbb{T})$ via $f = \sum_{n\in\mathbb{Z}} x_n z^n$. Hence, L becomes the multiplication operator $f \mapsto a \cdot f$, where

$$a(z) = -z^{-4} - (3+2\mathrm{i})z^{-3} + \mathrm{i}z^{-2} + z^{-1} + 10z + (3+\mathrm{i})z^2 + 4z^3 + \mathrm{i}z^4,$$

is the symbol of L. Arguing as we did in Exercise 1.2, we see that $\mathrm{Sp}(L) = \{a(z) : z \in \mathbb{T}\}$. Code for the second part of the exercise can be found in "chapter1/whale_example.m" in the repository.

Exercise 1.4

Suppose first that

$$z \in \bigcup_{E:\|E\|<\epsilon} \mathrm{Sp}(A+E)$$

and let E be such that $\|E\| < \epsilon$ and $z \in \mathrm{Sp}(A+E)$. Suppose, for a contradiction, that $\|(A - zI)^{-1}\| \leq \epsilon^{-1}$, which implies that $z \notin \mathrm{Sp}(A)$. Write:

$$A + E - zI = (I + E(A - zI)^{-1})(A - zI).$$

[1]If you have not come across this theorem, try to prove the result for the Hilbert space $\ell^2(\mathbb{N})$. This can be done 'component-wise' using the fact that a bounded sequence of complex numbers has a convergent subsequence.

Since $\|E\| < \epsilon$, we have $\|E(A - zI)^{-1}\| \leq \|E\|\|(A - zI)^{-1}\| < 1$, so by a standard Neumann series argument, $I+E(A-zI)^{-1}$ is invertible. Since $A - zI$ is invertible, we have that $A + E - zI$ is also invertible. This contradicts $z \in \mathrm{Sp}(A + E)$. So we have that $\|(A - zI)^{-1}\|^{-1} < \epsilon$.

Conversely, suppose that $z \notin \mathrm{Sp}(A)$ has $\|(A - zI)^{-1}\|^{-1} < \epsilon$. Then $\sigma_{\mathrm{inf}}(A - zI) = \|(A - zI)^{-1}\|^{-1} < \epsilon$. By definition of $\sigma_{\mathrm{inf}}(A - zI)$, there exists a unit-norm vector $x \in \mathcal{D}(A)$ such that $y = (A - zI)x$ has $\|y\| < \epsilon$. Define $Eu = -\langle u, x\rangle y$ for each $u \in \mathcal{H}$. Then $\|E\| = \|y\| < \epsilon$, while $(A - zI + E)x = 0$ with $x \neq 0$, so $z \in \mathrm{Sp}_{\mathrm{p}}(A + E) \subset \mathrm{Sp}(A + E)$.

In fact, we have shown that

$$\{z \in \mathbb{C} : \|(A - zI)^{-1}\|^{-1} < \epsilon\} = \bigcup_{E:\|E\|<\epsilon} \mathrm{Sp}(A + E) = \mathrm{Sp}(A) \cup \bigcup_{E:\|E\|<\epsilon} \mathrm{Sp}_{\mathrm{p}}(A + E).$$

Taking closures finishes the proof.

Exercise 1.5

We first consider the operator acting on $\mathrm{span}\{\cos(nt), \sin(nt) : n \in \mathbb{N}\}$. We have

$$\mathcal{T} \cos(nt) = \sin(nt), \quad \mathcal{T} \sin(nt) = -\cos(nt)$$

Direct integration yields

$$\frac{1}{4}\int_{-t}^{t} \cos(ns)\,\mathrm{d}s = \frac{\sin(nt)}{2n}, \quad \frac{1}{4}\int_{-t}^{t} \sin(ns)\,\mathrm{d}s = 0.$$

Mopping up the differential terms with direct differentiation, we see that

$$\mathcal{A}\begin{pmatrix}\cos(nt)/\sqrt{\pi}\\ \sin(nt)/\sqrt{\pi}\end{pmatrix} = A_n\begin{pmatrix}\cos(nt)/\sqrt{\pi}\\ \sin(nt)/\sqrt{\pi}\end{pmatrix}.$$

So at least formally on this basis, $\mathcal{A}$ has the matrix representation given by $A = \mathrm{diag}(A_1, A_2, \ldots)$. We formally define

$$\mathcal{D}(\mathcal{A}) = \left\{u(t) = \sum_{n=1}^{\infty} a_n \cos(nt) + b_n \sin(nt) : \sum_{n=1}^{\infty} |a_n|^2 + |b_n|^2 n^2 < \infty\right\}.$$

It is easy to check that $\mathcal{A}$ is closed and that the above basis forms a core.

Suppose now that $|z| < (\sqrt{5} - 1)/2$. Then

$$(A_n - zI)^{-1} = \frac{1}{z^2 - (2n + 1)}\begin{pmatrix}-z & -2n\\ -1 - \frac{1}{2n} & -z\end{pmatrix}$$

This has norm bounded above by (using the bound on $|z|$)

$$\frac{1}{(2n + 1) - |z|^2}\left\|\begin{pmatrix}-z & 0\\ 0 & -z\end{pmatrix}\right\| + \frac{1}{(2n + 1) - |z|^2}\left\|\begin{pmatrix}0 & -2n\\ -1 - \frac{1}{2n} & 0\end{pmatrix}\right\| = \frac{|z| + 2n}{(2n + 1) - |z|^2} \leq 1.$$

It follows that $\|(A - zI)^{-1}\| \leq 1$. However, as $n \to \infty$, $\|(A_n - zI)^{-1}\| \to 1$. Hence, we must have $\|(A - zI)^{-1}\| = 1$. This also proves that

$$\mathrm{Cl}\left(\left\{z \in \mathbb{C} : \left\|(A - zI)^{-1}\right\|^{-1} < 1\right\}\right) \neq \left\{z \in \mathbb{C} : \left\|(A - zI)^{-1}\right\|^{-1} \leq 1\right\}.$$

To see why

$$\mathbb{R}_{>0} \ni \epsilon \mapsto \mathrm{Sp}_{\epsilon}(A) \in \mathcal{M}_{\mathrm{AW}}$$

is not continuous, we take $\epsilon = 1$ and consider the above example. There is an open neighbourhood around $z = 0$ that lies in $\mathrm{Sp}_{\epsilon}(\mathcal{A})$ for every $\epsilon > 1$, but has empty intersection with $\mathrm{Sp}_1(\mathcal{A})$.

To prove that the map is left-continuous, let $\epsilon > 0$ and $\epsilon_n \uparrow \epsilon$. We will use the characterisation of convergence in the Attouch–Wets topology. Clearly $\mathrm{Sp}_{\epsilon_n}(A) \subset \mathrm{Sp}_{\epsilon}(A)$. Suppose for a contradiction that $\mathrm{Sp}_{\epsilon_n}(A)$ does not converge to $\mathrm{Sp}_{\epsilon}(A)$, then there exists some $z \in \mathrm{Sp}_{\epsilon}(A)$ and $\delta > 0$ such that (after taking a subsequence if necessary – which is unnecessary since the pseudospectra are nested!) $\mathrm{dist}(z, \mathrm{Sp}_{\epsilon_n}(A)) \geq \delta$. We may choose w with $\|(A - wI)^{-1}\|^{-1} < \epsilon$ and $|z - w| < \delta/2$. Since $\epsilon_n \uparrow \epsilon$, $\epsilon_n > \|(A - wI)^{-1}\|^{-1}$ so that $w \in \mathrm{Sp}_{\epsilon_n}(A)$ for large n, a contradiction.

Exercise 1.6

Suppose that $u : S \to \mathbb{R}$ is subharmonic and attains a maximum at $z \in S$. For every $r \geq 0$ such that $B_r(z) \subset S$,

$$u(z) \leq \frac{1}{2\pi}\int_0^{2\pi} u(z + re^{\mathrm{i}\theta})\,\mathrm{d}\theta \leq u(z).$$

Hence, $u(z + re^{\mathrm{i}\theta}) = u(z)$ for almost every θ. Since u is upper-semicontinuous, $u(z + re^{\mathrm{i}\theta}) = u(z)$ for all θ. Since $r \geq 0$ was arbitrary (subject to $B_r(z) \subset S$) , $u = u(z)$ on an open neighbourhood of z. Hence, the set of points w where $u(w) = u(z)$ is an open subset of S. It must be closed since the set where an upper-semicontinuous function attains its maximum is closed. Since S is connected, this set must be the whole of S and, hence, $u = u(z)$ on S.

We let S be a domain containing z_0 such that S is a subset of the resolvent set $\mathbb{C} \setminus \mathrm{Sp}(A)$ (which is open). Let $B_r(z) \subset S$. Direct integration yields that

$$(A - zI)^{-1} = \frac{1}{2\pi}\int_0^{2\pi} (A - (z + re^{\mathrm{i}\theta})I)^{-1}\,\mathrm{d}\theta.$$

Let $x, y \in \mathcal{H}$ with $\|x\|, \|y\| \leq 1$. Then,

$$\begin{aligned}\left|\langle (A - zI)^{-1}x, y\rangle\right| &= \left|\frac{1}{2\pi}\int_0^{2\pi} \langle (A - (z + re^{\mathrm{i}\theta})I)^{-1}x, y\rangle\,\mathrm{d}\theta\right| \\ &\leq \frac{1}{2\pi}\int_0^{2\pi} |\langle (A - (z + re^{\mathrm{i}\theta})I)^{-1}x, y\rangle|\,\mathrm{d}\theta \leq \frac{1}{2\pi}\int_0^{2\pi} \|(A - (z + re^{\mathrm{i}\theta})I)^{-1}\|\,\mathrm{d}\theta.\end{aligned}$$

Taking the supremum over such x and y on the left-hand side, we obtain

$$\|(A - zI)^{-1}\| \leq \frac{1}{2\pi}\int_0^{2\pi} \|(A - (z + re^{\mathrm{i}\theta})I)^{-1}\|\,\mathrm{d}\theta$$

and, hence, $z \mapsto \|(A - zI)^{-1}\|$ is subharmonic on S (it is continuous by Exercise 1.8).

For the final part, suppose that S is a bounded component of

$$\left\{z \in \mathbb{C} : \left\|(A - zI)^{-1}\right\|^{-1} < \epsilon\right\},$$

but that $S \cap \mathrm{Sp}(A) = \emptyset$. Then $z \mapsto \|(A - zI)^{-1}\|$ is subharmonic on S. Suppose that $z \in \mathrm{Cl}(S) \setminus S$, then we must have $\|(A - zI)^{-1}\|^{-1} = \epsilon$ by continuity of $z \mapsto \|(A - zI)^{-1}\|^{-1}$. Let $M = \sup_{z \in S} \|(A - zI)^{-1}\|$. Then

$$\epsilon^{-1} < M = \sup_{z \in \mathrm{Cl}(S)} \|(A - zI)^{-1}\| < \infty$$

since S is bounded and $z \mapsto \|(A - zI)^{-1}\|$ is continuous on $\mathrm{Cl}(S)$. So $\|(A - zI)^{-1}\|$ attains a maximum in S. It follows that $\|(A - zI)^{-1}\|$ is constant on S, which implies that it must equal ϵ^{-1} and, hence, $S = \emptyset$, the required contradiction.

Exercise 1.7

The map $z \mapsto \gamma(z, A) = \|(A - zI)^{-1}\|^{-1}$ is continuous on $\mathbb{C}$. We assume throughout this proof that it is not constant on an open subset of $\mathbb{C} \setminus \mathrm{Sp}(A)$, i.e., that the resolvent norm $\|(A - zI)^{-1}\|$ is not constant on an open subset of $\mathbb{C} \setminus \mathrm{Sp}(A)$.

Suppose, for a contradiction, that

$$\mathrm{Cl}\left(\{z \in \mathbb{C} : \gamma(z, A) < \epsilon\}\right) \neq \{z \in \mathbb{C} : \gamma(z, A) \leq \epsilon\} \quad \forall \epsilon > 0.$$

The set on the right-hand side is closed (preimage of a closed set under a continuous map) and, hence, contains the set on the left-hand side. It follows that there exists $z \in \mathbb{C}$ and $r > 0$ with $\gamma(z, A) = \epsilon$ and $\gamma(w, A) \geq \epsilon$ for all w with $|z - w| \leq r$. But this contradicts the maximum principle of the non-constant subharmonic function $z \mapsto \|(A - zI)^{-1}\|$.

We showed in Exercise 1.5 that the map $\mathbb{R}_{>0} \ni \epsilon \mapsto \mathrm{Sp}_\epsilon(A) \in \mathcal{M}_{\mathrm{AW}}$ is left-continuous. Hence, it is sufficient to show that it is right-continuous. Let $\epsilon_n \downarrow \epsilon > 0$. We wish to show that $\mathrm{Sp}_{\epsilon_n}(A)$ converges to $\mathrm{Sp}_\epsilon(A)$ in the Attouch–Wets topology as $n \to \infty$. We use the characterisation for this convergence (Exercise 1.10) to see that we must show that for each compact set K and $\delta > 0$, there exists N such that if $n \geq N$, then

$$\mathrm{Sp}_{\epsilon_n}(A) \cap K \subset \mathrm{Sp}_{\epsilon}(A) + B_\delta(0) \quad \text{and} \quad \mathrm{Sp}_{\epsilon}(A) \cap K \subset \mathrm{Sp}_{\epsilon_n}(A) + B_\delta(0).$$

The second inclusion always holds so suppose, for a contradiction, that the first does not for a specific K and δ. By sequential compactness, we may assume without loss of generality that there exists $z_n \in \mathrm{Sp}_{\epsilon_n}(A)$ with $z_n \to z$ and $\mathrm{dist}(z_n, \mathrm{Sp}_\epsilon(A)) \geq \delta$. Then $\gamma(z, A) = \lim_{n\to\infty} \gamma(z_n, A) \leq \lim_{n\to\infty} \epsilon_n = \epsilon$ so that $z \in \mathrm{Sp}_\epsilon(A)$ (from the first part of the exercise), the required contradiction.

Exercise 1.8

First note that $\|(A - zI)x\| \leq \|(A - wI)x\| + |z - w|$ for each $x \in \mathcal{D}(A)$ with $\|x\| = 1$ and $z, w \in \mathbb{C}$. Taking the infimum over x, we see that $z \mapsto \sigma_{\mathrm{inf}}(A - zI)$ is Lipschitz continuous with Lipschitz constant bounded by one. We can apply a similar argument to $z \mapsto \sigma_{\mathrm{inf}}(A^* - \bar{z}I)$. Since $\gamma(z, A) = \min\{\sigma_{\mathrm{inf}}(A - zI), \sigma_{\mathrm{inf}}(A^* - \bar{z}I)\} = \|(A - zI)^{-1}\|^{-1}$, the first part of the exercise follows.

Now suppose that A is a normal operator. If $z \in \mathrm{Sp}(A)$, then clearly $\mathrm{dist}(z, \mathrm{Sp}(A)) = \|(A - zI)^{-1}\|^{-1} = 0$. Hence, we may assume that $z \notin \mathrm{Sp}(A)$. Without loss of generality, by considering shifts by the identity, we may further assume that $z = 0$. We first claim that

$$\mathrm{Sp}(A^{-1}) \setminus \{0\} = \{\lambda^{-1} : \lambda \in \mathrm{Sp}(A)\}.$$

To see this, note that for every $w \neq 0$,

$$A^{-1} - wI = w(w^{-1}I - A)A^{-1}.$$

If $A^{-1} - wI$ is boundedly invertible then its kernel is trivial and it has range equal to $\mathcal{H}$. Hence, the range of $w^{-1}I - A$ is equal to $\mathcal{H}$. Moreover, if $x \in \ker(w^{-1}I - A)$, then $Ax \in \ker(A^{-1} - wI) = \{0\}$. Since A is invertible, $x = 0$. Hence, $(w^{-1}I - A)$ is boundedly invertible. Similarly, for every $w \neq 0$,

$$w^{-1}I - A = w^{-1}(A^{-1} - wI)A.$$

We can argue to see that if $w^{-1}I - A$ is boundedly invertible, then so is $A^{-1} - wI$. With this claim in hand, we use the fact that for a bounded normal operator B, $\|B\| = \sup_{z\in\mathrm{Sp}(B)} |z|$. It follows that

$$\|A^{-1}\| = \sup_{z\in\mathrm{Sp}(A^{-1})} |z| = \sup_{z\in\mathrm{Sp}(A^{-1})\setminus\{0\}} |z| = \sup_{\lambda\in\mathrm{Sp}(A)} \frac{1}{|\lambda|} = \frac{1}{\mathrm{dist}(0, \mathrm{Sp}(A))},$$

where the second equality uses the fact that $\mathrm{Sp}(A^{-1}) \setminus \{0\} \neq \emptyset$ by the above claim.

For the final part, note that we can now write

$$\mathrm{Sp}_\epsilon(A) = \mathrm{Cl}\left(\{z \in \mathbb{C} : \mathrm{dist}(z, \mathrm{Sp}(A)) < \epsilon\}\right) = \{z \in \mathbb{C} : \mathrm{dist}(z, \mathrm{Sp}(A)) \leq \epsilon\} = \mathrm{Sp}(A) + B_\epsilon(0).$$

NB: After you have read Chapter 4, you may wish to redo this exercise using the spectral theorem for normal operators.

Exercise 1.9

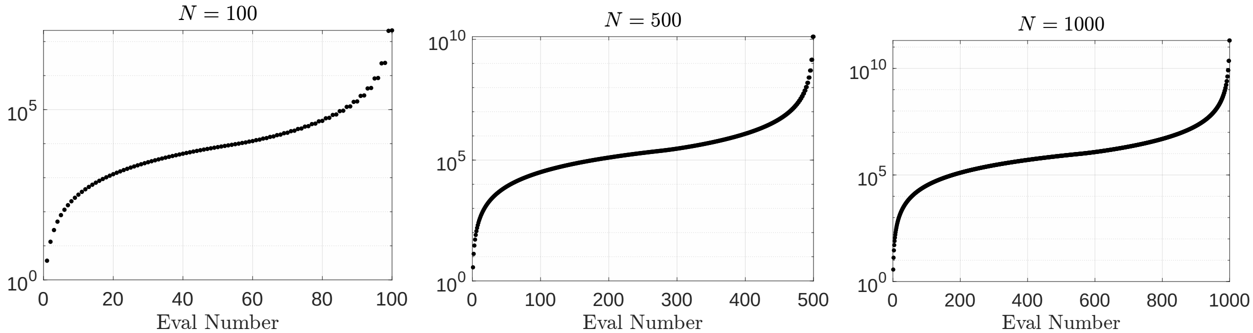
The infinite matrices can be recovered from the recurrence relations. Code for them and this exercise can be found in "ex1_9.m" in the repository. Suppose that $\mathcal{L}u = \lambda u$. Assuming that all expansions, summations and swapping of summations and integrals are valid, if $u = \sum_{j=1}^\infty u_j f_j$, then, using the orthogonality of Legendre Polynomials,

$$\langle \mathcal{L}u, P_{i-1}\rangle = \|P_{i-1}\|^2 \sum_{j=1}^\infty A_{ij}u_j, \quad \langle u, P_{i-1}\rangle = \|P_{i-1}\|^2 \sum_{j=1}^\infty B_{ij}u_j.$$

Hence, we formally have

$$\sum_{j=1}^\infty A_{ij}u_j = \lambda \sum_{j=1}^\infty B_{ij}u_j, \quad i = 1, 2, \ldots,$$

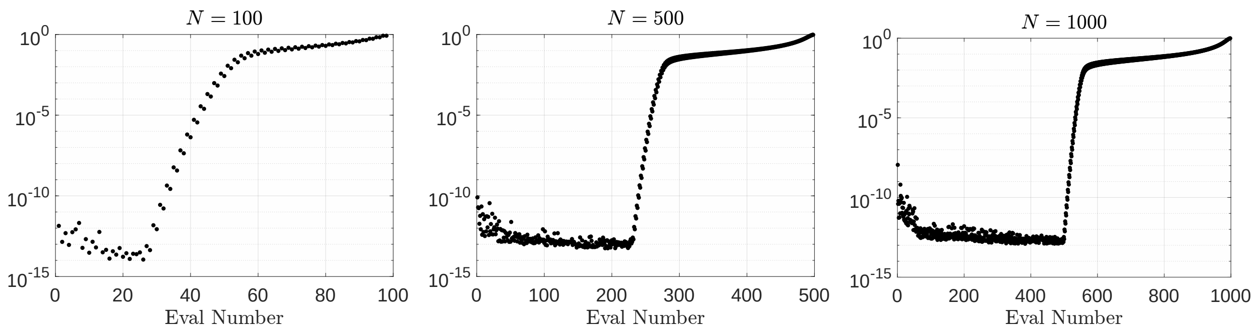
i.e., the generalised eigenvalue problem $A - \lambda B$. Here are eigenvalues for various $N \times N$ discretisations:



Since $\mathcal{L}$ is self-adjoint, Exercise 1.8 implies that

$$\operatorname{dist}(\lambda, \operatorname{Sp}(\mathcal{L})) = \sigma_{\inf}(\mathcal{L} - \lambda I) \leq \|\mathcal{L}v - \lambda v\| / \|v\|.$$

Here are these residuals, normalised by the size of the computed eigenvalues to produce approximations of relative errors:



We see that there is a sharp phase transition from high accuracy to low accuracy, occurring at a constant proportion of the number of computed eigenvalues.

Exercise 1.10

Throughout this solution, let $C, C_n \subset \mathbb{C}$ be compact, non-empty sets.

We first prove the characterisation in the lemma. Clearly, all the conditions at the end of the lemma are equivalent. Suppose first that $\lim_{n\to\infty} d_{\mathrm{AW}}(C_n, C) = 0$ and $K \subset \mathbb{C}$ is compact. Choose M large so that $K \subset B_M(0)$. If $a \in C_n \cap K$, then $\operatorname{dist}(a, C_n) = 0$. Hence,

$$\sup_{a \in C_n \cap K} \operatorname{dist}(a, C) = \sup_{a \in C_n \cap K} |\operatorname{dist}(a, C_n) - \operatorname{dist}(a, C)| \leq \sup_{|z| \leq M} |\operatorname{dist}(z, C_n) - \operatorname{dist}(z, C)|.$$

The right-hand side must converge to 0 as $n \to \infty$ by definition of d_{AW}. We can argue similarly to see that $\sup_{a \in C \cap K} \operatorname{dist}(a, C_n)$ must converge to 0. Hence, $\lim_{n\to\infty} d_K(C_n, C) = 0$.

Now suppose that for every compact $K \subset \mathbb{C}$, $\lim_{n\to\infty} d_K(C_n, C) = 0$. Let $m \in \mathbb{N}$ be sufficiently large so that $C \cap B_m(0) \neq \emptyset$. Let $z \in C \cap B_m(0)$. Then $\operatorname{dist}(z, C_n) \leq d_{B_m(0)}(C_n, C)$ and, hence, $\lim_{n\to\infty} \operatorname{dist}(z, C_n) = 0$. By making m larger if necessary, and by ignoring an initial part of the sequence $\{C_n\}$, we may assume without loss of generality that $C_n \cap B_m(0) \neq \emptyset$ for all n. For $\epsilon > 0$, we may choose N so that if $n \geq N$, then $C \cap B_{4m}(0) \subset C_n + B_\epsilon(0)$ and $C_n \cap B_{4m}(0) \subset C + B_\epsilon(0)$. If $|z| \leq m$, then there exists $w \in C \cap B_{4m}(0)$ with $\operatorname{dist}(z, C) = |z - w|$. Hence,

$$\operatorname{dist}(z, C_n) \leq \operatorname{dist}(z, C_n + B_\epsilon(0)) + \epsilon \leq \operatorname{dist}(z, C \cap B_{4m}(0)) + \epsilon \leq |z - w| + \epsilon = \operatorname{dist}(z, C) + \epsilon.$$

Swapping the roles of C and C_n, we see that $\operatorname{dist}(z, C) \leq \operatorname{dist}(z, C_n) + \epsilon$. Since $\epsilon > 0$ was arbitrary,

$$\lim_{n\to\infty} \sup_{|z| \leq m} |\operatorname{dist}(z, C_n) - \operatorname{dist}(z, C)| = 0.$$

We can argue for larger m similarly. Using a simple argument to bound the tail of the series defining d_{AW} by a geometric series, we see that $\lim_{n\to\infty} d_{\mathrm{AW}}(C_n, C) = 0$. Hence, the lemma is proven.

Suppose that $\lim_{n\to\infty} d_{\mathrm{H}}(C_n, C) = 0$. Let $K \subset \mathbb{C}$ be compact. Then,

$$d_K(C_n, C) = \max\left\{\sup_{a\in C_n\cap K} \mathrm{dist}(a, C), \sup_{b\in C\cap K} \mathrm{dist}(b, C_n)\right\} \le \max\left\{\sup_{a\in C_n} \mathrm{dist}(a, C), \sup_{b\in C} \mathrm{dist}(b, C_n)\right\} = d_{\mathrm{H}}(C_n, C).$$

Hence, by the characterisation proven above, $d_{\mathrm{AW}}(C_n, C) \to 0$ as $n \to \infty$.

Now suppose that $\cup_n C_n$ is contained in some bounded ball $B_R(0)$ and $\lim_{n\to\infty} d_{\mathrm{AW}}(C_n, C) = 0$. We may choose a compact set $K \subset \mathbb{C}$ that contains C and every C_n. Hence, for this choice of K,

$$d_{\mathrm{H}}(C_n, C) = \max\left\{\sup_{a\in C_n} \mathrm{dist}(a, C), \sup_{b\in C} \mathrm{dist}(b, C_n)\right\} = \max\left\{\sup_{a\in C_n\cap K} \mathrm{dist}(a, C), \sup_{b\in C\cap K} \mathrm{dist}(b, C_n)\right\} = d_K(C_n, C).$$

Hence, by the characterisation proven above, $d_{\mathrm{H}}(C_n, C) \to 0$ as $n \to \infty$.

We cannot drop the assumption of $\cup_n C_n$ being in some bounded ball $B_R(0)$. For example, let $C_n = \{0, n\}$ and $C = \{0\}$. Then $d_{\mathrm{AW}}(C_n, C) \to 0$ as $n \to \infty$ but $d_{\mathrm{H}}(C_n, C) = n$ diverges.

Exercise 1.11

Let A be a closed, densely defined operator on $\mathcal{H}$. Suppose, for a contradiction, that $\mathrm{Sp}_\epsilon(A)$ does not converge to $\mathrm{Sp}(A)$ in the Attouch–Wets topology as $\epsilon \downarrow 0$. Using the characterisation of the Attouch–Wets topology and the fact that $\mathrm{Sp}(A) \subset \mathrm{Sp}_\epsilon(A)$ for every $\epsilon > 0$, it follows that there exists $\epsilon_n \downarrow 0$, $\delta > 0$ and $z_n \in \mathrm{Sp}_{\epsilon_n}(A) \cap K$ for some compact K such that $\mathrm{dist}(z_n, \mathrm{Sp}(A)) \ge \delta$. Since K is compact, we may assume without loss of generality that $\lim_{n\to\infty} z_n = z$. By continuity of $w \mapsto \|(A - wI)^{-1}\|^{-1}$, $\|(A - zI)^{-1}\|^{-1} = 0$ and, hence, $z \in \mathrm{Sp}(A)$, the required contradiction.

Exercise 1.12

The first part is immediate from the definition of σ_{inf}.

For the example, let A be the shift operator on $\ell^2(\mathbb{N})$ given by $Ae_1 = 0$ and $Ae_n = e_{n-1}$ for $n = 2, 3, \ldots$. Then $\sigma_{\mathrm{inf}}(A) = 0$. However, the adjoint of A is an isometry and, hence, $\sigma_{\mathrm{inf}}(A^*) = 1$.

Suppose now that $\mathrm{Sp}(A)$ has empty interior. For all $z \notin \mathrm{Sp}(A)$, $\sigma_{\mathrm{inf}}(A - zI) = \sigma_{\mathrm{inf}}(A^* - \bar{z}I)$, so assume that $z \in \mathrm{Sp}(A)$. Then there exists $z_n \in \mathbb{C} \setminus \mathrm{Sp}(A)$ with $\lim_{n\to\infty} z_n = z$. Hence,

$$\sigma_{\mathrm{inf}}(A - zI) = \lim_{n\to\infty} \sigma_{\mathrm{inf}}(A - z_n I) = \lim_{n\to\infty} \sigma_{\mathrm{inf}}(A^* - \overline{z_n} I) = \sigma_{\mathrm{inf}}(A^* - \bar{z}I).$$

It follows that $\mathrm{Sp}(A) = \mathrm{Sp}_{\mathrm{ap}}(A)$ and we always have $\sigma_{\mathrm{inf}}(A - zI) = \sigma_{\mathrm{inf}}(A^* - \bar{z}I)$.

Exercise 1.13

For the first part, suppose that $A - zI$ is a bijection. The inverse mapping theorem (which follows from the open mapping theorem – see any standard proof) implies that $A - zI$ has a bounded inverse and, hence, $z \notin \mathrm{Sp}(A)$.

If z is in the point spectrum, then clearly $\sigma_{\mathrm{inf}}(A - zI) = 0$. Suppose that z is in the continuous spectrum (the definition given in the exercise) but, for a contradiction, $\sigma_{\mathrm{inf}}(A - zI) > 0$. If $\{y_n = (A - zI)x_n\}$ is Cauchy, then since $\|x_n - x_m\| \le [\sigma_{\mathrm{inf}}(A - zI)]^{-1}\|y_n - y_m\|$, $\{x_n\}$ is Cauchy and converges. Since A is bounded, we see that the limit $y = \lim_{n\to\infty} y_n$ lies in the range of $(A - zI)$. Hence, this range must be closed. It is dense and, hence, must be the whole of $\mathcal{H}$ so that $A - zI$ is a bijection. But this contradicts $z \in \mathrm{Sp}(A)$.

The converse is not true. Take $\mathcal{H} = \ell^2(\mathbb{N})$ and consider A defined by $Ae_n = n^{-1}e_{n+1}$. This is injective and does not have dense range (e_1 cannot be in the closure of the range of A). Hence, 0 lies in the residual spectrum of A. However, $\lim_{n\to\infty} Ae_n = 0$ so that $\sigma_{\mathrm{inf}}(A) = 0$.

Exercise 1.14

For $x \in \mathcal{D}(A)$ with $\|x\| = 1$ write

$$\|(A_n - zI)x\| \le \|(A_n - A)x\| + \|(A - zI)x\|$$

so that

$$\sigma_{\inf}(A_n - zI) \leq \|(A_n - A)x\| + \|(A - zI)x\| \quad \forall n \in \mathbb{N}.$$

We have $\|(A_n - A)x\| \to 0$, so taking lim sup on both sides we have:

$$\limsup_{n\to\infty} \sigma_{\inf}(A_n - zI) \leq \|(A - zI)x\|.$$

Taking the infimum over x, we get the first result. For the example, let A_n be defined on $\ell^2(\mathbb{N})$ by $A_n^* e_m = e_{n+m}$ and $A = 0$. Then $\lim_{n\to\infty} \|A_n v - Av\| = 0$ for every $v \in \ell^2(\mathbb{N})$. However, $\sigma_{\inf}(A^*) = 0$ and $\sigma_{\inf}(A_n^*) = 1$.

Exercise 1.15

By Lemma 1.2.6, γ_{n_2,n_1} is increasing in n_1 so that

$$\lim_{n_1\to\infty} \Gamma_{n_3,n_2,n_1}(A) = \Gamma_{n_3,n_2}(A) = \left\{ z \in \mathrm{Grid}(n_2) : \gamma_{n_2}(z, A) + \frac{1}{n_2} \leq \frac{1}{n_3} \right\}.$$

It is here that using '$\leq$' in the definition of $\Gamma_{n_3,n_2,n_1}(A)$ is crucial. Since $\gamma_{n_2}(z, A) \geq \gamma(z, A)$, the '$+1/n_2$' term ensures that

$$\Gamma_{n_3,n_2}(A) \subset \left\{ z \in \mathbb{C} : \gamma(z, A) < \frac{1}{n_3} \right\} \subset \mathrm{Sp}_{\frac{1}{n_3}}(A).$$

This care is needed since the resolvent norm of an unbounded operator can be constant on open sets. Suppose, for a contradiction, that $\Gamma_{n_3,n_2}(A)$ does not converge to $\mathrm{Sp}_{\frac{1}{n_3}}(A)$ in the Attouch–Wets topology as $n_2 \to \infty$. By the characterisation in Lemma 1.2.4, there exists a compact set K, $z_{n_2} \in \mathrm{Sp}_{\frac{1}{n_3}}(A) \cap K$, and $\delta > 0$ such that

$$\limsup_{n_2\to\infty} \mathrm{dist}(z_{n_2}, \Gamma_{n_3,n_2}(A)) \geq \delta.$$

Without loss of generality, we may assume that $\lim_{n_2\to\infty} z_{n_2} = z \in \mathrm{Sp}_{\frac{1}{n_3}}(A) \cap K$ by closedness. We may choose $w \in \mathbb{C}$ with $|z - w| \leq \delta/2$ and $\gamma(w, A) < 1/n_3$. Now let w_{n_2} be an element of $\mathrm{Grid}(n_2)$ nearest to w. Then, since $\gamma_{n_2}(z, A)$ is Lipschitz with Lipschitz constant bounded by one,

$$\gamma_{n_2}(w_{n_2}, A) + \frac{1}{n_2} \leq \gamma_{n_2}(w, A) + |w - w_{n_2}| + \frac{1}{n_2}.$$

The bound on the right-hand side is smaller than $1/n_3$ for large n_2 so that $w_{n_2} \in \Gamma_{n_3,n_2}(A)$. Moreover, $\lim_{n_2\to\infty} w_{n_2} = w$ and, hence, $|z - w_{n_2}| < \delta$ for large n_2, the required contradiction. To get the third limit we use convergence of $\mathrm{Sp}_\epsilon(A)$ to $\mathrm{Sp}(A)$ as $\epsilon \downarrow 0$.

For the final part and to make this completely rigorous, all we need to do is replace γ_{n_2,n_1} by an approximation that is increasing in n_1 and so that the error of the approximation vanishes as $n_1 \to \infty$. We can easily do this by approximating γ_{n_2,n_1} from below.

Exercise 1.16

Let A be a bounded operator with n-dimensional range $\mathrm{ran}(A)$. By choosing an orthonormal basis of $\mathrm{ran}(A)$, we see that $\{Ax : \|x\| \leq 1\}$ is homeomorphic to a closed subset of $\{z \in \mathbb{C}^n : \|z\| \leq \|A\|\}$, which is compact. Hence, A is compact.

Suppose now that A is compact and let $\{e_n : n \in \mathbb{N}\}$ be an orthonormal basis of $\mathcal{H}$ with $\mathcal{P}_n : \mathcal{H} \to \mathrm{span}\{e_1, \ldots, e_n\}$ denoting the corresponding orthogonal projection. We first prove that $B_n = \mathcal{P}_n^* \mathcal{P}_n A$ converges to A in the operator norm topology. It is easily seen that B_n converges in the strong operator topology to A. Given $\epsilon > 0$, there exists a finite set $\{x_1, \ldots, x_m\}$ such that

$$\{Ax : \|x\| \leq 1\} \subset \bigcup_{j=1}^{m} \{y : \|y - Ax_j\| \leq \epsilon\}.$$

We may choose N so that if $n \geq N$, then $\sup_{j=1,\ldots,m} \|(A - B_n)x_j\| \leq \epsilon$. Given x with $\|x\| \leq 1$, choose x_j such that $\|B_n(x - x_j)\| \leq \|A(x - x_j)\| \leq \epsilon$. Then for $n \geq N$,

$$\|(A - B_n)x\| \leq \|A(x - x_j)\| + \|Ax_j - B_n x_j\| + \|B_n(x - x_j)\| \leq 3\epsilon.$$

Since $\epsilon > 0$ was arbitrary, $\lim_{n\to\infty} \|A - B_n\| = 0$. We can apply the same argument to the adjoint of A (A^* is compact by Schauder's theorem) to see that $B\mathcal{P}_n^*\mathcal{P}_n$ converges to B in the operator norm topology for each compact operator B. Given $\delta > 0$, choose N such that $\|A - B_N\| \leq \delta$. Since B_N is compact,

$$\limsup_{n\to\infty} \|A - B_N\mathcal{P}_n^*\mathcal{P}_n\| \leq \delta + \limsup_{n\to\infty} \|B_N - B_N\mathcal{P}_n^*\mathcal{P}_n\| = \delta.$$

For $n \geq N$, $\|A - A_n\| \leq \|A - B_N\mathcal{P}_n^*\mathcal{P}_n\|$, and hence the required convergence holds since $\delta > 0$ was arbitrary.

Exercise 1.17

Recall that for a bounded operator C, if $\|C\| < 1$ then (by a Neumann series argument) $I - C$ is invertible and $\|(I - C)^{-1}\| \leq (1 - \|C\|)^{-1}$. This is a standard argument that can be seen by proving that

$$I + C + C^2 + \cdots = (I - C)^{-1}.$$

In the setting of the exercise, let $C = -BA^{-1}$, which is bounded with $\|C\| \leq \|B\|\|A^{-1}\| < 1$. Let $S = (I - C)^{-1}$, then

$$\|S\| \leq \frac{1}{1 - \|B\|\|A^{-1}\|}.$$

Let $D = A^{-1}S$, then D is a bounded operator and

$$(A + B)D = (I + BA^{-1})S = I, \quad D(A + B) = A^{-1}S(I + BA^{-1})A = A^{-1}A \subset I.$$

Hence, $A + B$ is boundedly invertible with inverse D and

$$\|(A + B)^{-1}\| \leq \|A^{-1}\|\|S\| \leq \frac{\|A^{-1}\|}{1 - \|B\|\|A^{-1}\|}.$$

Suppose that for some $\epsilon > 0$ there exists a sequence $\{B_n\}_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} \|B_n\| = 0$ and

$$\sup_{z\in\mathrm{Sp}(A+B_n)} \mathrm{dist}(z, \mathrm{Sp}(A)) \geq \epsilon.$$

Select $z_n \in \mathrm{Sp}(A + B_n)$ such that $\mathrm{dist}(z_n, \mathrm{Sp}(A)) \geq \epsilon$. If $\{z_n\}_{n\in\mathbb{N}}$ is bounded, then we have that $\|(A - z_nI)^{-1}\|$ is bounded in n by continuity and the fact that $\{z_n\}_{n\in\mathbb{N}}$ lies a positive distance from $\mathrm{Sp}(A)$. Otherwise, passing to a subsequence we have $|z_n| \to \infty$ and $\|(A - z_nI)^{-1}\| \to 0$ since A is bounded. So either way, passing to a subsequence if necessary, we have $\|B_n\|\|(A - z_nI)^{-1}\| \to 0$ as $n \to \infty$. Then $0 \notin \mathrm{Sp}(A + B_n - z_nI)$ by the first part of the exercise and, hence, $z_n \notin \mathrm{Sp}(A + B_n)$ for large enough n, a contradiction.

Exercise 1.18

Suppose for a contradiction that the statement is false. By taking a subsequence if necessary, we may assume that there exists $\epsilon > 0$ such that

$$\inf_{z\in\mathrm{Sp}(A+B_n)} \mathrm{dist}(z, X) \geq \epsilon \quad \forall n \in \mathbb{N}. \tag{1}$$

We may also assume that X is at least 2ϵ separated from the rest of the spectrum of A. Let $S = X + D_{\epsilon/2}(0)$, which is open and non-empty. We first prove that S is connected.

Suppose for a contradiction that S is not connected. Then $S = U \cup V$, where U and V are non-empty disjoint (relatively open and, hence, since S is open,) open subsets of $\mathbb{C}$. Then $X \cap U$ and $X \cap V$ are both relatively open in X and disjoint. Since X is connected, we may assume without loss of generality that $X \subset U$. Take an arbitrary $z \in V$. Then $z \notin U$ and hence $\mathrm{dist}(z, X) \in (0, \epsilon/2)$. Let $x \in X$ be a nearest point in X to z (exists as connected components are closed) and set

$$w(t) = (1 - t)z + tx, \quad t \in [0, 1].$$

Then $w(t) \in S$ for such t. Let

$$t_* = \inf\{t \in [0, 1] : w(t) \in U\}.$$

Then $t_* \in (0,1)$. Since U and V are both open, it is easily seen that $w(t_*) \notin U \cup V$, the required contradiction.

It follows that S is a domain so that we may apply Exercise 1.6. In particular, the bound in (1) implies that the functions $z \mapsto \|(A + B_n - zI)^{-1}\|$ are subharmonic on S. Now take an arbitrary point $z \in X$. Note that $0 \notin \mathrm{Sp}(A + B_n - zI)$. If $\|B_n\|\|(A + B_n - zI)^{-1}\| < 1$, then Exercise 1.17 implies that $0 \notin \mathrm{Sp}(A - zI)$, a contradiction. Hence, $\|(A + B_n - zI)^{-1}\| \geq \|B_n\|^{-1}$, which grows unbounded as $n \to \infty$. By the maximum principle, there exists $z_n \in \mathrm{Cl}(S) \setminus S$ such that $\lim_{n\to\infty} \|(A + B_n - z_n I)^{-1}\| = \infty$. Since $\mathrm{Cl}(S)$ is compact, we may assume that $\lim_{n\to\infty} z_n = w$ without loss of generality. Moreover, we must have that $\mathrm{dist}(w, X) = \epsilon/2$. For sufficiently large n, $\|B_n + (w - z_n)I\|\|(A - wI)^{-1}\| < 1/2$, so that we may apply Exercise 1.17 once more to see that

$$\|(A + B_n - z_n I)^{-1}\| \leq \frac{\|(A - wI)^{-1}\|}{1 - \|B_n + (w - z_n)I\|\|(A - wI)^{-1}\|} \leq 2\|(A - wI)^{-1}\|.$$

This is a clear contradiction as $\lim_{n\to\infty} \|(A + B_n - z_n I)^{-1}\| = \infty$ and, hence, our assumption in (1) must be false.

Exercise 1.19

Suppose first that $\epsilon = 0$. Then A_0 acts as the unilateral shift $e_n \mapsto e_{n+1}$ on the invariant subspace $\mathrm{Cl}(\mathrm{span}\{e_n : n \in \mathbb{N}\})$. Similarly, after a relabelling, it acts as the transpose of such a unilateral shift on $\mathrm{Cl}(\mathrm{span}\{e_n : n \in \mathbb{Z}_{\leq 0}\})$. It follows that $\mathrm{Sp}(A_0) = \{z : |z| \leq 1\}$ as this is the spectrum of such a unilateral shift. If $\epsilon > 0$, we define the basis

$$\hat{e}_n = e_n \text{ if } n \geq 1 \text{ and } \hat{e}_n = \epsilon^{-1} e_n \text{ if } n \leq 0.$$

The matrix representation of A_ϵ with respect to this basis is a matrix with 1s along the subdiagonal and zeros elsewhere. It follows that A_ϵ is related to the bilateral shift $e_n \mapsto e_{n+1}$ by a bounded similarity transformation with bounded inverse. Hence, its spectrum is the same as the bilateral shift, which is $\{z : |z| = 1\}$. This example shows that the spectrum is discontinuous as a map from the set of bounded operators with the operator norm topology to the set of non-empty closed subsets of $\mathbb{C}$ with the Hausdorff topology (in contrast to what happens in the self-adjoint case).

Exercise 1.20

As in Exercise 1.16, we set $A_n = \mathcal{P}_n^*\mathcal{P}_n A \mathcal{P}_n^*\mathcal{P}_n$. If $z \in \mathrm{Sp}(A) \setminus \{0\}$, then it must be isolated, and, hence, a bounded component of the spectrum separated from the rest of the spectrum. Exercise 1.16 and Exercise 1.18 immediately imply that $\mathrm{dist}(z, \mathrm{Sp}(A_n))$ converges to zero as $n \to \infty$. We may decompose $\ell^2(\mathbb{N})$ into $\mathrm{span}\{e_1, \ldots, e_n\}$ and its orthogonal complement and write

$$\mathcal{P}_n^*\mathcal{P}_n A \mathcal{P}_n^*\mathcal{P}_n = \begin{pmatrix} \mathcal{P}_n A \mathcal{P}_n^* & \\ & 0 \end{pmatrix}, \quad (\mathcal{P}_n^*\mathcal{P}_n A \mathcal{P}_n^*\mathcal{P}_n - zI)^{-1} = \begin{pmatrix} (\mathcal{P}_n A \mathcal{P}_n^* - zI)^{-1} & \\ & -1/z \end{pmatrix}.$$

It follows that $\mathrm{Sp}(A_n) = \mathrm{Sp}(\mathcal{P}_n A \mathcal{P}_n^*) \cup \{0\}$ and

$$\lim_{n\to\infty} \mathrm{dist}(z, \mathrm{Sp}(\mathcal{P}_n A \mathcal{P}_n^*)) = 0.$$

There are now two cases to consider. In the first scenario, there are infinitely many distinct nonzero eigenvalues of A that accumulate at 0. In this case, we may apply the above argument to a sequence of such eigenvalues converging to zero to see that

$$\lim_{n\to\infty} \mathrm{dist}(0, \mathrm{Sp}(\mathcal{P}_n A \mathcal{P}_n^*)) = 0.$$

In the second scenario, A has only finitely many nonzero eigenvalues. It is easily shown that $\lim_{n\to\infty} \sigma_{\mathrm{inf}}(\mathcal{P}_n A \mathcal{P}_n^*) = 0$. Moreover, $X = \{0\}$ is an isolated point of the spectrum of A. Hence, we can adapt the argument of the proof in the answer to Exercise 1.18 to see that

$$\lim_{n\to\infty} \mathrm{dist}(0, \mathrm{Sp}(\mathcal{P}_n A \mathcal{P}_n^*)) = 0.$$

This completes the first part of the exercise.

Suppose now for a contradiction that

$$\limsup_{n\to\infty} \sup_{z \in \mathrm{Sp}(\mathcal{P}_n A \mathcal{P}_n^*)} \mathrm{dist}(z, \mathrm{Sp}(A)) \neq 0.$$

By choosing a subsequence if necessary, we may assume that there exists $\epsilon > 0$ and $z_n \in \mathrm{Sp}(\mathcal{P}_n A \mathcal{P}_n^*)$ with $\mathrm{dist}(z_n, \mathrm{Sp}(A)) \geq \epsilon$. Since the spectra of $\mathcal{P}_n A \mathcal{P}_n^*$ are uniformly bounded, we may assume that $\lim_{n\to\infty} z_n = z$. Note that $z_n \in \mathrm{Sp}(A_n)$. We set $B_n = A_n - A + (z - z_n)I$. Since $0 \notin \mathrm{Sp}(A - zI)$ and $\lim_{n\to\infty} \|B_n\| = 0$, by Exercise 1.17, we see that $0 \notin \mathrm{Sp}(A + B_n - zI)$ for sufficiently large n. But $A + B_n - zI = A_n - z_n I$, a contradiction.

Exercise 1.21

Suppose that $z \in \mathbb{C} \setminus \mathrm{Sp}(A)$, $(A - zI)^{-1}$ is compact, and $w \in \mathbb{C} \setminus \mathrm{Sp}(A)$. We may write

$$(A - wI)^{-1} = (A - wI)^{-1} [(A - zI) - (A - wI)] (A - zI)^{-1} + (A - zI)^{-1} = [(w - z)(A - wI)^{-1} + I](A - zI)^{-1}.$$

The operator on the right-hand side is compact since it is the product of a bounded and compact operator.

Suppose now that A has compact resolvent with $(A - zI)^{-1}$ compact. Let $w \notin \mathrm{Sp}((A - zI)^{-1})$. Since $w \neq 0$,

$$(A - zI)^{-1} - wI = (I - w(A - zI))(A - zI)^{-1} = -w[A - (w^{-1} + z)I](A - zI)^{-1}.$$

Let $S = -w(A - zI)^{-1}((A - zI)^{-1} - wI)^{-1}$. Then S is bounded and $[A - (w^{-1} + z)I]S = I$. A similar argument shows that $S[A - (w^{-1} + z)I] \subset I$. Hence,

$$\mathrm{Sp}(A) \subset \{w^{-1} + z : w \in \mathrm{Sp}((A - zI)^{-1}) \setminus \{0\}\},$$

which can only consist of isolated points. Now if $w \in \mathrm{Sp}((A - zI)^{-1}) \setminus \{0\}$, then it must be an eigenvalue of $(A - zI)^{-1}$ with eigenvector v. It is easily checked that v is an eigenvector of A with eigenvalue $w^{-1} + z$. Hence, the spectrum of A consists of isolated points that are eigenvalues.

For the final part, consider the following (unbounded) tridiagonal operator:

$$A = \bigoplus_{n=1}^{\infty} \begin{pmatrix} 0 & n \\ n & 0 \end{pmatrix},$$

where we take the closure of this operator with initial domain consisting of vectors that are finite linear combinations of the canonical basis functions. Then A is self-adjoint and has compact resolvent with $\mathrm{Sp}(A) = \{\pm k : k \in \mathbb{N}\}$. However, $0 \in \mathrm{Sp}(\mathcal{P}_n A \mathcal{P}_n^*)$ for n odd.

Exercise 1.22

Suppose for a contradiction that a Σ_1-algorithm $\{\Gamma_n\}_{n\in\mathbb{N}}$ exists for the spectral problem on self-adjoint compact operators. (Σ_1 corresponds to the first type of error control in the statement of the theorem.) Since $d_{\mathrm{H}}(\Gamma_n(C), \{0, 1\}) \to 0$, $\Gamma_n(C)$ intersects $B_{1/4}(1)$ for sufficiently large $n \geq 3$. Consider the computation of $\Gamma_n(C)$: this will use finitely many of the matrix entries $\langle Ce_j, e_i\rangle$, which are a subset of $\{i, j \leq N(C, n)\}$ for some integer $N(C, n)$. Take $k > N(C, n)$ and set $A = A_k$. The matrix elements of A_k differ from C only at the positions $(1, k)$, $(k, 1)$, (k, k), which means that by consistency of Γ_n, we must have $\Gamma_n(A) = \Gamma_n(C)$. From the Σ_1 classification, we have $\Gamma_n(A) \subset \mathrm{Sp}(A) + B_{2^{-n}}(0)$ and so $\Gamma_n(C) \subset \{0, 2\} + B_{2^{-n}}(0) = B_{2^{-n}}(0) \cup B_{2^{-n}}(2)$. However $\Gamma_n(C)$ intersects $B_{1/4}(1)$, which is disjoint from both $B_{2^{-n}}(0)$ and $B_{2^{-n}}(2)$ since $n \geq 3$. We therefore have a contradiction.

The case of Π_1 is similar. Suppose that $\{\Gamma_n\}_{n\in\mathbb{N}}$ is a Π_1-algorithm for the spectral problem on self-adjoint compact operators. (Π_1 corresponds to the second type of error control in the statement of the theorem.) Since $d_{\mathrm{H}}(\Gamma_n(C), \{0, 1\}) \to 0$, $\Gamma_n(C)$ is disjoint from $B_{1/4}(2)$ for sufficiently large $n \geq 3$. Defining $N(C, n)$ as before, take $k > N(C, n)$. As before we have $\Gamma_n(C) = \Gamma_n(A)$. Then we have $\mathrm{Sp}(A) \subset \Gamma_n(C) + B_{2^{-n}}(0)$. Then $2 \in \Gamma_n(C) + B_{2^{-n}}(0)$, implying that $\Gamma_n(C)$ intersects $B_{2^{-n}}(2)$. This is contrary to our initial assumption, and we have a contradiction.

Recall that for A self-adjoint, we have $\mathrm{Sp}_\epsilon(A) = \mathrm{Sp}(A) + B_\epsilon(0)$. If $\{\Gamma_n\}_{n\in\mathbb{N}}$ solves the pseudospectral problem for $\epsilon < 1/4$, then:

1. $\Gamma_n(C)$ eventually intersects $B_{1/4+\epsilon}(1)$, which is disjoint from $B_{2^{-n}+\epsilon}(0) \cup B_{2^{-n}+\epsilon}(2)$
2. $\Gamma_n(C)$ is eventually disjoint from $B_{1/4+\epsilon}(2)$, while $2 \in \mathrm{Sp}_\epsilon(A)$.

The statements for pseudospectra then follow easily from proof of the statements for spectra. We can rescale the operators and argument to deal with arbitrary $\epsilon > 0$.

The matrix elements of A^2 correspond to knowing $\langle Ae_i, Ae_j\rangle$ for each $i, j \in \mathbb{N}$. Since A is self-adjoint we also know $\langle A^* e_i, A^* e_j\rangle$. By operator folding (see Section 3.4), we can compute arbitrarily good approximations of $\gamma_n(z, A)$. We can use these in the Σ_1^A algorithm for spectra in Section 3.2.3 since $\mathrm{dist}(z, \mathrm{Sp}(A)) = \|(A - zI)^{-1}\|^{-1}$.

2 Chapter 2

Exercise 2.1

The idea is that we can pass to a subsequence depending on A. Let $\{\Xi, \Omega, \mathcal{M}, \Lambda\}$ be a computational problem and suppose that $\{\Gamma_n\}_{n\in\mathbb{N}}$ is a sequence of algorithms of type α that satisfy the following: if $A \in \Omega$, for each $n \in \mathbb{N}$ there exists $E_n(A)$ that we can compute such that $d(\Gamma_n(A), \Xi(A)) \leq E_n(A)$ and $E_n(A) \to 0$ as $n \to \infty$. Let $A \in \Omega$ and construct an increasing sequence as follows: let $n_1 = n_1(A)$ be least such that $E_{n_1}(A) \leq 2^{-1}$ (by simply keeping on checking $E_n(A)$ for increasing n until $E_n(A) \leq 2^{-1}$), and for $k \geq 1$ let $n_{k+1}(A) > n_k(A)$ be least such that $E_{n_{k+1}}(A) \leq 2^{-(k+1)}$. Taking $\widetilde{\Gamma}_k(A) = \Gamma_{n_k(A)}(A)$, we have $d(\widetilde{\Gamma}_k(A), \Xi(A)) \leq 2^{-k}$, and, hence, $\{\widetilde{\Gamma}_k\}_{k\in\mathbb{N}}$ is a Δ_1^α algorithm for $\{\Omega, \Lambda, \mathcal{M}, \Xi\}$. The other classes and cases are similar.

Exercise 2.2

Let $\epsilon \in \mathbb{Q}_{>0}$ with $\epsilon < 1$ and set $\delta = \epsilon/2$. For each $A \in \mathbb{C}^{m\times m}$, we can compute using finitely many arithmetic operations and comparisons a function $\nu(z, A)$ with

$$\gamma(z, A) \leq \nu(z, A) \leq \gamma(z, A) + \delta.$$

We can also compute $R \in \mathbb{N}$ such that $\|A\| \leq R - 1$ (e.g., by bounding the Frobenius norm of A from above). Let

$$G_\epsilon = \{z \in \delta(\mathbb{Z} + \mathrm{i}\mathbb{Z}) : |z| \leq R\}, \quad \Gamma_\epsilon(A) = \{z \in G_\epsilon : \nu(z, A) \leq \epsilon\}.$$

Clearly, $\Gamma_\epsilon(A) \subset \mathrm{Sp}_\epsilon(A)$. Hence, if $z \in \Gamma_\epsilon(A)$, then

$$\mathrm{dist}(z, \mathrm{Sp}(A)) \leq \mathrm{dist}(z, \mathrm{Sp}_\epsilon(A)) + d_{\mathrm{H}}(\mathrm{Sp}_\epsilon(A), \mathrm{Sp}(A)) \leq F(\epsilon).$$

If $z \in \mathrm{Sp}(A)$, then there exists $w \in G_\epsilon$ that minimises $|z - w| \leq \delta$ and, hence, by the Lipschitz property of γ,

$$\nu(w, A) \leq \gamma(w, A) + \delta \leq \gamma(z, A) + 2\delta = 2\delta = \epsilon.$$

It follows that $w \in \Gamma_\epsilon(A)$ and, hence, $\mathrm{dist}(z, \Gamma_\epsilon(A)) \leq \epsilon$. Hence, $d_{\mathrm{H}}(\Gamma_\epsilon(A), \mathrm{Sp}(A)) \leq \max\{\epsilon, F(\epsilon)\}$. We now consider a sequence of decreasing values for ϵ and apply Exercise 2.1.

Exercise 2.3

Throughout this proof, the algorithm type α is either arithmetic or general.

Let $\{\Xi, \Omega, \mathcal{M}_{\mathrm{dec}}, \Lambda\}$ be a computational problem and suppose that $\{\Gamma_n\}_{n\in\mathbb{N}}$ is a sequence of algorithms of type α with $\lim_{n\to\infty} \Gamma_n(A) = \Xi(A)$ for all $A \in \Omega$. Then, for every $A \in \Omega$,

$$\Xi(A) = 1 \quad \Leftrightarrow \quad \forall n \exists k (k \geq n \wedge \Gamma_k(A)) \quad \Leftrightarrow \quad \exists n \forall k (k \leq n \vee \Gamma_k(A)).$$

Similarly, suppose that $\{\Gamma_n\}_{n\in\mathbb{N}}$ are algorithms of type α and let $\varphi : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$, $k \mapsto (\varphi_1(k), \varphi_2(k))$ be a bijection which enumerates all pairs of natural numbers. Then

$$\exists n \exists m (\Gamma_{n,m}(A)) \Leftrightarrow \exists k (\Gamma_{\varphi_1(k),\varphi_2(k)}(A)), \qquad \forall n \forall m (\Gamma_{n,m}(A)) \Leftrightarrow \forall k (\Gamma_{\varphi_1(k),\varphi_2(k)}(A)).$$

Thus, every limit in a tower of height m can be converted alternately into an expression with two quantifiers ($\forall\exists$ or $\exists\forall$), and then $m - 1$ doubles $\exists\exists$ or $\forall\forall$ can be replaced by single quantifiers.

If $\{\Xi, \Omega, \mathcal{M}_{\mathrm{dec}}, \Lambda\} \in \Sigma_m^\alpha$, we may replace the final limit (corresponding to $\exists\forall$) with $\exists$. For example, when $m = 3$, we obtain

$$\exists|\exists\forall|\forall\exists,$$

where the | separate limits. This can be further simplified to

$$\exists\forall\exists.$$

This argument can be carried out to any number of limits. Hence, we obtain an alternating quantifier form of length m with the first quantifier equal to $\exists$. Conversely, if such a form exists it is easy to see that $\{\Xi, \Omega, \mathcal{M}_{\mathrm{dec}}, \Lambda\} \in \Sigma_m^\alpha$. This proves (i). Swapping the role of $\exists$ and $\forall$, we argue in the same manner to prove (ii).

Suppose that $\mathrm{SCI}(\Xi, \Omega, \mathcal{M}_{\mathrm{dec}}, \Lambda)_\alpha \leq m$, then we may apply the first part of the solution to get an alternating quantifier form of length $m+1$ with first quantifier equal to $\exists$. Hence, $\{\Xi, \Omega, \mathcal{M}_{\mathrm{dec}}, \Lambda\} \in \Sigma_{m+1}^\alpha$ by part (i). We can do the same with an alternating quantifier form of length $m+1$ with first quantifier equal to $\forall$ to see that $\{\Xi, \Omega, \mathcal{M}_{\mathrm{dec}}, \Lambda\} \in \Pi_{m+1}^\alpha$.

Conversely, suppose that $\{\Xi, \Omega, \mathcal{M}_{\mathrm{dec}}, \Lambda\} \in \Sigma_{m+1}^\alpha \cap \Pi_{m+1}^\alpha$. We must show that $\mathrm{SCI}(\Xi, \Omega, \mathcal{M}_{\mathrm{dec}}, \Lambda)_\alpha \leq m$, which will finish the proof of (iii). In the above steps we have already seen that $\mathrm{SCI}(\Xi, \Omega, \mathcal{M}_{\mathrm{dec}}, \Lambda)_\alpha \leq m+1$, and we next prove the following: Suppose that

$$\Xi(A) = \exists i \forall j (g_0(i, j, A)) = \forall p \exists q (g_1(p, q, A)),$$

where g_0 and g_1 are algorithms of type α. Then $\Xi(A) = \lim_{k\to\infty} g(k, A)$ with a function g being easily derivable from g_0, g_1. Fix A and define a function $h_0 : \mathbb{N} \to \mathcal{M}_{\mathrm{dec}}$ recursively as follows:

$i(1) := 1$, $j(1) := 1$, $h_0(1) := g_0(i(1), j(1), A)$.
`If` $h_0(l) = 1$
`then:` $i(l+1) := i(l)$, $j(l+1) := j(l) + 1$
`else:` $i(l+1) := i(l) + 1$, $j(l+1) := 1$.
$l := l + 1$.
$h_0(l) := g_0(i(l), j(l), A)$.

We observe that, if $\Xi(A) = 1$ then $h_0(l)$ converges as $l \to \infty$ with limit 1. Otherwise, the limit does not exist or is 0. The same construction applies to $\neg(\forall p \exists q (g_1(p, q, A))) = \exists p \forall q \neg(g_1(p, q, A))$ and yields a function h_1 which converges to 1 if and only if $\Xi(A) = 0$. Clearly, exactly one of the functions h_0, h_1 converges to 1. Now we derive the desired g from h_0 and h_1 as follows:

$\alpha(1) = 0$.
`If` $h_{\alpha(k)}(k) = 1$
`then:` $\alpha(k+1) := \alpha(k)$
`else:` $\alpha(k+1) := 1 - \alpha(k)$.
$k := k + 1$.
`If` $\alpha(k) = 0$
`then:` $g(k, A) := 1$
`else:` $g(k, A) := 0$.

This provides $\Xi(A) = \lim_{k\to\infty} g(k, A)$.

Next, let g_0 and g_1 be of the form $g_s(i, j, A) = \lim_{r\to\infty} f_{i,j,r}^s(A)$, $s \in \{0, 1\}$, where s is an index and not a power. Fix A. Then for every pair (i, j) there is an $r(i, j)$ such that $f_{u,v,r}^s(A) = g_s(u, v, A)$ for all $u \leq i$, $v \leq j$, $s \in \{0, 1\}$ and $r \geq r(i, j)$. Thus, g is also of the form $g(k, A) = \lim_{r\to\infty} f_{k,r}(A)$ with $f_{k,r}$ being defined by the above procedure applied to the functions $(i, j, A) \mapsto f_{i,j,k}^s(A)$ instead of $g_s(i, j, A)$ $(s \in \{0, 1\})$.

Now we are left with iterating this argument: If both functions g_s $(s \in \{0, 1\})$ are of the form $g_s(i, j, A) = \lim_{k_{m-1}\to\infty} \lim_{k_{m-2}\to\infty} \cdots \lim_{k_1\to\infty} f_{i,j,k_{m-1},\cdots,k_1}^s(A)$ with certain type-α algorithms $f_{i,j,k_{m-1},\cdots,k_1}^s$, then also g is of the form

$$g(k, A) = \lim_{k_{m-1}\to\infty} \lim_{k_{m-2}\to\infty} \cdots \lim_{k_1\to\infty} f_{k,k_{m-1},\cdots,k_1}(A)$$

with $f_{k,k_{m-1},\cdots,k_1}$ being defined by the same procedure as before applied to the functions $(i, j, A) \mapsto f_{i,j,k_{m-1},\cdots,k_1}^s(A)$ instead of $g_s(i, j, A)$ $(s \in \{0, 1\})$. The resulting functions $A \mapsto f_{k,k_{m-1},\cdots,k_1}(A)$ are type-α algorithms for every k, since their evaluation requires only finitely many evaluations of the type-α algorithms $f_{i,j,k_{m-1},\cdots,k_1}^s$. Moreover, we have

$$\Xi(A) = \lim_{k\to\infty} g(k, A) = \lim_{k\to\infty} \lim_{k_{m-1}\to\infty} \cdots \lim_{k_1\to\infty} f_{k,k_{m-1},\cdots,k_1}(A),$$

so that $\mathrm{SCI}(\Xi, \Omega, \mathcal{M}_{\mathrm{dec}}, \Lambda)_\alpha \leq m$.

The above arguments can be extended to other types of algorithms. The only requirement is that they are closed under the various operations and comparisons above.

Exercise 2.4

Let $\{\Omega, \Lambda, \mathcal{M}, \Xi\}$ be a computational problem. Suppose that we have constructed a type-α tower $\{\Gamma_n\}_{n\in\mathbb{N}}$ such that $\Gamma_n(A) \subset \Xi(A) + B_{E_n(A)}(0)$ for each $n \in \mathbb{N}$ and $A \in \Omega$ with $E_n(A)$ given by the algorithm. By passing to a subsequence depending on A, as in Exercise 2.1, without loss of generality we assume that $E_n(A) \leq 2^{-n}$ (then, since E_n is just an upper bound, we can take $E_n(A) = 2^{-n}$). Let

$$X_n(A) = \Xi(A) \cup \Gamma_n(A).$$

Then $\Gamma_n(A) \subset X_n(A)$, we just need to show that $d_{\mathrm{H}}(X_n(A), \Xi(A)) \leq 2^{-n}$. Since $\Gamma_n(A) \subset \Xi(A) + B_{2^{-n}}(0)$, we have $X_n(A) \subset \Xi(A) + B_{2^{-n}}(0)$. Also we have $\Xi(A) \subset X_n(A)$, so we also have $\Xi(A) \subset X_n(A) + B_{2^{-n}}(0)$. We therefore obtain $d_{\mathrm{H}}(X_n(A), \Xi(A)) \leq 2^{-n}$. Note that this does not give a Δ_1^α algorithm since $\Xi(A)$ is not accessible.

Similarly, for the Π_1^α classification, it suffices to construct a type-α tower $\{\Gamma_n\}_{n\in\mathbb{N}}$ with a provided error $\{E_n(A)\}_{n\in\mathbb{N}}$ such that $\Xi(A) \subset \Gamma_n(A) + B_{E_n(A)}(0)$ for each $n \in \mathbb{N}$. Again, we can without loss of generality take $E_n(A) = 2^{-n}$. Take $X_n(A) = \Gamma_n(A) \cup \Xi(A)$. Then $\Xi(A) \subset X_n(A)$, so we just need to show that $d_{\mathrm{H}}(X_n(A), \Gamma_n(A)) \leq 2^{-n}$. Since $\Xi(A) \subset \Gamma_n(A) + B_{2^{-n}}(0)$ and $\Gamma_n(A) \subset \Gamma_n(A) + B_{2^{-n}}(0)$, we get $X_n(A) \subset \Gamma_n(A) + B_{2^{-n}}(0)$. Likewise, since $\Gamma_n(A) \subset X_n(A)$ we get $\Gamma_n(A) \subset X_n(A) + B_{2^{-n}}(0)$. So we get $d_{\mathrm{H}}(\Gamma_n(A), X_n(A)) \leq 2^{-n}$.

We can generalise these as follows for $m \geq 2$:

- Σ_m^α: it is enough to construct a type-α tower $\{\Gamma_{n_m,\ldots,n_1}\}_{(n_m,\ldots,n_1)\in\mathbb{N}^m}$ such that:

$$\lim_{n_m\to\infty} \ldots \lim_{n_1\to\infty} \Gamma_{n_m,\ldots,n_1}(A) = \Xi(A)$$

  for each $A \in \Omega$ with accessible $\{E_{n_m}(A)\}_{n_m\in\mathbb{N}}$ such that $\Gamma_{n_m}(A) \subset \Xi(A) + B_{E_{n_m}(A)}(0)$ for each $n_m \in \mathbb{N}$.

- Π_m^α: it is enough to construct a type-α tower $\{\Gamma_{n_m,\ldots,n_1}\}_{(n_m,\ldots,n_1)\in\mathbb{N}^m}$ such that:

$$\lim_{n_m\to\infty} \ldots \lim_{n_1\to\infty} \Gamma_{n_m,\ldots,n_1}(A) = \Xi(A)$$

  for each $A \in \Omega$ with accessible $\{E_{n_m}(A)\}_{n_m\in\mathbb{N}}$ such that $\Xi(A) \subset \Gamma_{n_m}(A) + B_{E_{n_m}(A)}(0)$ for each $n_m \in \mathbb{N}$.

The proofs are the same as above, simply considering the last limit.

Exercise 2.5

Define a weighted norm by

$$\|x\|_w^2 = \sum_{n=1}^{\infty} n^2 |x_n|^2, \quad \text{and set} \quad \mathcal{D} = \left\{ \{x_n\}_{n\in\mathbb{N}} \in \ell^2(\mathbb{N}) : \|x\|_w < \infty \right\},$$

which is dense in $\ell^2(\mathbb{N})$ since it contains the eventually zero sequences. On $\mathcal{D}$, define the operators

$$T_n = \begin{pmatrix} 1 & & & & 1 \\ & 2 & & & \\ & & \ddots & & \\ & & & n & \\ 1 & & & & 1 \end{pmatrix} \oplus \mathrm{diag}(n+1, n+2, \ldots) \text{ for } n \geq 2 \quad \text{and} \quad S = \mathrm{diag}(1, 2, 3, \ldots).$$

Both T_n and S are closed, and $\mathrm{span}\{e_n\}$ forms a core of both operators. We have $\mathrm{Sp}(T_n) = \{0, 2, 3, \ldots\}$ for each $n \in \mathbb{N}$ and $\mathrm{Sp}(S) = \{1, 2, 3, \ldots\}$.

We first show that S is self-adjoint. Clearly S is symmetric so that $\mathcal{D} \subset \mathcal{D}(S^*)$. It remains to show that $\mathcal{D}(S^*) \subset \mathcal{D}$. Let $y \in \mathcal{D}(S^*)$. Define $y^{(k)}$ by $y_n^{(k)} = n y_n$ for $n \leq k$ and $y_n^{(k)} = 0$ for $n > k$. Write $e^{(k)} = \frac{y^{(k)}}{\|y^{(k)}\|}$. By definition of the domain of the adjoint, $x \mapsto \langle Sx, y\rangle$ is a linear functional bounded by some constant M. This implies that $|\langle S e^{(k)}, y\rangle| \leq M$ for each k. Hence,

$$|\langle S e^{(k)}, y\rangle| = \sum_{n=1}^{k} n e_n^{(k)} \overline{y_n} = \frac{\sum_{n=1}^{k} n^2 |y_n|^2}{\|y^{(k)}\|} = \sqrt{\sum_{n=1}^{k} n^2 |y_n|^2} \leq M.$$

Taking $k \to \infty$ we find that $y \in \mathcal{D}$. Since T_n differs from S by a bounded operator, it is also self-adjoint. Clearly, S is positive. Note that

$$\langle T_n x, x\rangle = (x_1 + x_{n+1})\overline{x_1} + \sum_{k=2}^{n} k|x_k|^2 + (x_1 + x_{n+1})\overline{x_{n+1}} + \sum_{k\geq n+2} (k-1)|x_k|^2$$

for $x \in \mathcal{D}$. Then noticing that $(x_1 + x_{n+1})\overline{x_1} + (x_1 + x_{n+1})\overline{x_{n+1}} = |x_1 + x_{n+1}|^2$, we can see that $\langle T_n x, x\rangle \geq 0$ for each $x \in \mathcal{D}$ and hence T_n is positive (and, hence, semi-bounded below). Finally, we check that both T_n and S have compact resolvent. It can be seen that:

$$(T_n - I)^{-1} = \begin{pmatrix} 0 & & & & 1 \\ & 1 & & & \\ & & \ddots & & \\ & & & \frac{1}{n-1} & \\ 1 & & & & 0 \end{pmatrix} \oplus \operatorname{diag}\left(\frac{1}{n}, \frac{1}{n+1}, \frac{1}{n+2}, \ldots\right).$$

This is the norm limit of the finite rank operators

$$A_n^{(k)} = \begin{pmatrix} 0 & & & & 1 \\ & 1 & & & \\ & & \ddots & & \\ & & & \frac{1}{n-1} & \\ 1 & & & & 0 \end{pmatrix} \oplus \operatorname{diag}\left(\frac{1}{n+l} : 0 \leq l \leq k\right) \oplus 0,$$

and, hence, is compact. Similarly, one can check that S^{-1} is compact.

With everything checked, we move on to the proof. Suppose that $\{\Gamma_n\}_{n\in\mathbb{N}}$ realises a Σ_1^G classification for this spectral problem. Then $\Gamma_n(S)$ intersects the ball $B_{1/4}(1)$ for sufficiently large $n \geq 3$. Define $N(S, n)$ analogously as before in Exercise 1.22 and take $k > N(S, n)$ and $T = T_k$ so that $\Gamma_n(S) = \Gamma_n(T)$. Obtain $X_n(T) \supseteq \Gamma_n(T) = \Gamma_n(S)$ such that:

$$X_n(T) \cap B_2(0) \subset \operatorname{Sp}(T) + B_{2^{-2}}(0)$$

giving:

$$\Gamma_n(S) \cap B_2(0) \subset B_{2^{-2}}(0) \cup \bigcup_{j=2}^{\infty} B_{2^{-2}}(j),$$

so that:

$$\Gamma_n(S) \cap B_2(0) \subset B_{2^{-2}}(0) \cup B_{2^{-2}}(2).$$

By our choice of n, $\Gamma_n(S) \cap B_2(0)$ intersects $B_{1/4}(1)$, yet the right-hand side of this inclusion does not, contradiction.

The Π_1^G case is very similar. Suppose that $\{\Gamma_n\}_{n\in\mathbb{N}}$ realises a Π_1^G classification. We have that $\Gamma_n(S)$ is disjoint from $B_{3/4}(0)$ for sufficiently large n. Construct $N(S, n)$ as before and take $k > N(S, n)$ and $T = T_k$. Then $\Gamma_n(S) = \Gamma_n(T)$. Reasoning as before, we have:

$$\{0, 2\} \subset \Gamma_n(T) + B_{2^{-1}}(0) = \Gamma_n(S) + B_{2^{-1}}(0)$$

However, $\Gamma_n(S)$ does not intersect $B_{3/4}(0)$, so we cannot have $0 \in \Gamma_n(S) + B_{2^{-1}}(0)$, contradiction.

Exercise 2.6

We first apply the answer to Exercise 1.15, but replacing $1/n_3$ with ϵ and only taking the first two limits. This shows that $\{\operatorname{Sp}_\epsilon, \Omega_{l^2(\mathbb{N})}, \mathcal{M}_{\mathrm{AW}}, \Lambda\} \in \Sigma_2^A$ since the set obtained after the first limit lies in the pseudospectrum (a little care is needed to show the existence of arithmetic towers, but this is easily done following the surrounding discussion of rectangular truncations in Chapter 3). The proof that the computational problem is not in Δ_2^G is a simple adaptation of the proof of Theorem 3.1.3.

We can also apply another argument that uses scaling of pseudospectra. Suppose that $\{\operatorname{Sp}_{\epsilon_n}, \Omega_{l^2(\mathbb{N})}, \mathcal{M}_{\mathrm{AW}}, \Lambda\} \in \Delta_2^G$ for some sequence $\epsilon_n \to 0$, where $\{\Gamma_{n_1}^{\epsilon_{n_2}}\}$ are general algorithms with

$$\lim_{n_1\to\infty} \Gamma_{n_1}^{\epsilon_{n_2}}(A) = \operatorname{Sp}_{\epsilon_{n_2}}(A) \quad \forall A \in \Omega_{l^2(\mathbb{N})}.$$

Then $\Gamma_{n_2,n_1} = \Gamma_{n_1}^{\epsilon_{n_2}}$ defines a height-two tower of general algorithms for $\{\operatorname{Sp}, \Omega_{l^2(\mathbb{N})}, \mathcal{M}_{\mathrm{AW}}, \Lambda\}$ (using the fact that $\lim_{\epsilon\downarrow 0} \operatorname{Sp}_\epsilon(A) = \operatorname{Sp}(A)$), contradicting Theorem 2.3.10. Hence, for sufficiently small ϵ, we have $\{\operatorname{Sp}_\epsilon, \Omega_{l^2(\mathbb{N})}, \mathcal{M}_{\mathrm{AW}}, \Lambda\} \notin \Delta_2^G$. We may suppose that this lower bound holds for $\epsilon \leq \epsilon_0$ with $\epsilon_0 > 0$. Now suppose for a contradiction that $\{\Gamma_n\}$ is a height-one tower of general algorithms for $\{\mathrm{Sp}_\eta, \Omega_{l^2(\mathbb{N})}, \mathcal{M}_{\mathrm{AW}}, \Lambda\}$ for some $\eta > 0$. Let $c > 0$. Note that $z \in \mathrm{Sp}_\eta(cA)$ if and only if $z/c \in \mathrm{Sp}_{\eta/c}(A)$. We choose $c = \eta/\epsilon_0$. It follows that

$$\tilde{\Gamma}_n(A) = \frac{1}{c} \cdot \Gamma_n(cA)$$

provides a height-one tower of general algorithms for $\{\mathrm{Sp}_{\epsilon_0}, \Omega_{l^2(\mathbb{N})}, \mathcal{M}_{\mathrm{AW}}, \Lambda\}$, a contradiction.

Exercise 2.7

For the upper bounds, the key point is the observation that

$$\sigma_{\inf}((A - zI)\mathcal{P}_n^*) = \sqrt{\sigma_{\inf}(\mathcal{P}_n(A - zI)^*(A - zI)\mathcal{P}_n^*)}, \quad \sigma_{\inf}((A^* - \bar{z}I)\mathcal{P}_n^*) = \sqrt{\sigma_{\inf}(\mathcal{P}_n(A^* - \bar{z}I)^*(A^* - \bar{z}I)\mathcal{P}_n^*)}.$$

Hence, we can approximate $\gamma_n(z, A)$ to any desired accuracy using $\tilde{\Lambda}$ and finitely many arithmetic operations and comparisons. We can now avoid the first limit in the solution of Exercise 1.15 to obtain $\{\mathrm{Sp}, \Omega, \mathcal{M}, \tilde{\Lambda}\} \in \Pi_2^A$ and $\{\mathrm{Sp}_\epsilon, \Omega, \mathcal{M}, \tilde{\Lambda}\} \in \Sigma_1^A$.

The lower bound $\{\mathrm{Sp}_\epsilon, \Omega, \mathcal{M}, \tilde{\Lambda}\} \notin \Delta_1^G$ follows from the discussion of diagonal operators in Chapter 1 and observing that $\tilde{\Lambda}$ does not provide more information than entry evaluations on diagonal inputs. The lower bound $\{\mathrm{Sp}, \Omega, \mathcal{M}, \tilde{\Lambda}\} \notin \Delta_2^G$ follows from Theorem 3.1.2. The key point is that for a tridiagonal A, we have

$$\langle Ae_j, Ae_i\rangle = \left\langle Ae_j, \sum_{n=-1}^{1} \langle Ae_i, e_{i+n}\rangle e_{i+n} \right\rangle = \sum_{n=-1}^{1} \overline{\langle Ae_i, e_{i+n}\rangle} \langle Ae_j, e_{i+n}\rangle,$$

with a similar formula holding for $\langle A^*e_j, A^*e_i\rangle$. Hence, $\tilde{\Lambda}$ does not provide more information than entry evaluations on tridiagonal inputs, and we may directly apply Theorem 3.1.2.

Exercise 2.8

We recall the computational problems from Section 2.3.3. For $k \in \mathbb{Z}_{\geq 2}$, let Ω_k be the collection of all infinite arrays $a = \{a_{m_1,\ldots,m_k}\}_{m_1,\ldots,m_k \in \mathbb{N}}$ with entries $a_{m_1,\ldots,m_k} \in \{0, 1\}$ and Λ_k be the set of component-wise evaluation functions. We 'freeze' the indices $m_1, \ldots, m_{k-2}$ and consider the formulas

$$P(a, m_1, \ldots, m_{k-2}) = \begin{cases} 1, & \text{if } \exists i \; \forall j \; \exists n > j \text{ s.t. } a_{m_1,\ldots,m_{k-2},n,i} = 1, \\ 0, & \text{otherwise,} \end{cases}$$

$$Q(a, m_1, \ldots, m_{k-2}) = \begin{cases} 1, & \text{if } \forall^\infty i \forall j \; \exists n > j \text{ s.t. } a_{m_1,\ldots,m_{k-2},n,i} = 1, \\ 0, & \text{otherwise,} \end{cases}$$

where $\forall^\infty$ means 'for all but a finite number of'. P decides whether the corresponding matrix has a column with infinitely many 1's, whereas Q decides whether the matrix has only finitely many columns with only finitely many 1's. For $R = P$ or Q, consider the problem function for $a \in \Omega_k$

$$\Xi_{k,R}(a) = \begin{cases} \exists m_1 \; \forall m_2 \; \ldots \; \forall m_{k-2} R(a, m_1, \ldots, m_{k-2}), & \text{if } k \text{ is even,} \\ \forall m_1 \; \exists m_2 \; \ldots \; \forall m_{k-2} R(a, m_1, \ldots, m_{k-2}), & \text{otherwise,} \end{cases}$$

with alternating quantifiers $\exists$ and $\forall$ in front of the formula R. Theorem 2.3.9 says the following. Let $(\mathcal{M}, d)$ be $\{0, 1\}$ with the discrete metric or $[0, 1]$ with the usual metric and consider the above problems $\{\Xi_{k,R}, \Omega_k, \mathcal{M}, \Lambda_k\}$. For $k \in \mathbb{N}_{\geq 2}$ and $R = P$ or Q,

$$\Delta_{k+1}^G \not\ni \{\Xi_{k,R}, \Omega_k, \mathcal{M}, \Lambda_k\} \in \Delta_{k+2}^A.$$

The idea is to simultaneously embed all of these problems into a computational problem.

Let $\Omega = \prod_{k \geq 2} \Omega_k$ and Λ consist of the maps

$$(\{a_{m_{1,1},m_{1,2}}\}, \{a_{m_{2,1},m_{2,2},m_{2,3}}\}, \{a_{m_{3,1},m_{3,2},m_{3,3},m_{3,4}}\}, \ldots) \mapsto a_c, \quad c \in \bigcup_{k=2}^{\infty} \mathbb{N}^k.$$

In other words, Λ corresponds to the evaluation maps Λ_k on each Ω_k. We let $\mathcal{M} = [0,1]$ with the standard metric. For $A \in \Omega$, we can consider the decision problem $\Xi_{k,R}$ acting on its component in Ω_k. We then set

$$\Xi(A) = \sum_{k=2}^{\infty} 3^{-k} \Xi_{k,R}(A).$$

Suppose that $\{\Gamma_{n_p,\dots,n_1}\}_{(n_p,\dots,n_1)\in\mathbb{N}^p}$ is a height-p general tower of algorithms for $\{\Omega, \Lambda, \mathcal{M}, \Xi\}$. Pick arbitrary $b_k \in \Omega_k$ for $k \neq p+1$. For $a \in \Omega_{p+1}$, consider the input $A = (b_2, \dots, b_p, a, b_{p+2}, \dots)$. It follows that the $(p+1)$th ternary digit of $\Gamma_{n_p,\dots,n_1}(A)$ converges to $\Xi_{p+1,R}(A)$ in p limits. Moreover, the evaluation set Λ collapses to Λ_{p+1} since the b_k were fixed. Hence, we get a solution to $\{\Xi_{p+1,R}, \Omega_{p+1}, \{0,1\}, \Lambda_{p+1}\}$ in p limits, a contradiction.

Exercise 2.9

We use the setup of the solution to Exercise 2.8. For $n \in \mathbb{N}_{\geq 2}$, let $\delta_n \in (2^{-(n+1)}, 2^{-n})$. For $A \in \Omega$, computing $\Xi(A)$ to accuracy δ_n is equivalent to solving $\{\Xi_{k,R}, \Omega_k, \{0,1\}, \Lambda_k\}$ for $2 \leq k \leq n$, which is in $\Delta^A_{n+2} \setminus \Delta^G_{n+1}$. By taking the product with problems in $\Delta^A_2 \setminus \Delta^G_1$ and $\Delta^A_3 \setminus \Delta^G_2$, we can alter the construction in Exercise 2.8 to define δ_0 and δ_1 with $\delta_0 > \delta_1 > \delta_2$ so that computing $\Xi(A)$ to accuracy δ_n is in $\Delta^A_{n+2} \setminus \Delta^G_{n+1}$ for $n = 0, 1, 2, \dots$. A simple rescaling argument of the sum definition of Ξ allows us to alter and relabel the δ_n's so that $\Delta^G_n \not\ni \{\Xi^{(\epsilon_n)}, \Omega, \mathcal{M}, \Lambda\} \in \Delta^A_{n+1}$ for all $n \in \mathbb{N}$.

Exercise 2.10

We first prove the Lemma. Suppose first that $(\mathcal{M}, d)$ is the Hausdorff metric. If $x \in C$ then $x \in B'$ and $\mathrm{dist}(x, A) \leq d(B', A) \leq d(A, B) + \epsilon$. On the other hand, if $x \in A$ then $x \in A'$ and $\mathrm{dist}(x, C) \leq d(A', C) \leq \epsilon$. The result now follows. Suppose now that $(\mathcal{M}, d)$ is the Attouch–Wets metric and let $x \in \mathcal{M}'$. Since $C \subset_{\mathcal{M}'} B'$ we must have

$$\mathrm{dist}(x, A) - \mathrm{dist}(x, C) \leq \mathrm{dist}(x, A) - \mathrm{dist}(x, B') \leq |\mathrm{dist}(x, A) - \mathrm{dist}(x, B)| + |\mathrm{dist}(x, B) - \mathrm{dist}(x, B')|.$$

Similarly, since $A \subset_{\mathcal{M}'} A'$ we must have

$$\mathrm{dist}(x, C) - \mathrm{dist}(x, A) \leq \mathrm{dist}(x, C) - \mathrm{dist}(x, A') \leq |\mathrm{dist}(x, C) - \mathrm{dist}(x, A')|.$$

It follows that

$$|\mathrm{dist}(x, A) - \mathrm{dist}(x, C)| \leq |\mathrm{dist}(x, A) - \mathrm{dist}(x, B)| + |\mathrm{dist}(x, B) - \mathrm{dist}(x, B')| + |\mathrm{dist}(x, C) - \mathrm{dist}(x, A')|$$

and this finishes the proof of the Lemma. To prove Theorem 2.2.11, we will use the following proposition.

Proposition 2.1. *Let $(\mathcal{M}, d)$ be either a metric space with the Attouch–Wets or Hausdorff topology induced by another metric space $(\mathcal{M}', d_{\mathcal{M}'})$ or a totally ordered metric space with order respecting metric. Suppose we have a computational problem $\Xi : \Omega \to \mathcal{M}$, with a corresponding Σ^α_k-tower $\{\Gamma^1_{n_k,\dots,n_1}\}$ and a corresponding Π^α_k-tower $\{\Gamma^2_{n_k,\dots,n_1}\}$ (either both arithmetic or both general). Suppose also that $1 \leq k \leq 3$ and that, in the case of arithmetic towers, we can compute for every $A \in \Omega$ the distance $d(\Gamma^1_{n_k,\dots,n_1}(A), \Gamma^2_{n_k,\dots,n_1}(A))$ to arbitrary precision using finitely many arithmetic operations and comparisons. Then $\{\Xi, \Omega, \mathcal{M}, \Lambda\} \in \Delta^\alpha_k$.*

Proof. For $k = 1$, this is a trivial consequence of the lemma we just proved. Let δ_{n_1} be an approximation of

$$d(\Gamma^1_{n_1}(A), \Gamma^2_{n_1}(A)) + 2 \cdot 2^{-n_1}$$

from above to accuracy $1/n_1$. Note that suitable approximations can easily be generated using approximations of $d(\Gamma^1_{n_1}(A), \Gamma^2_{n_1}(A))$. Let $\epsilon > 0$, then simply choose $n_1 \in \mathbb{N}$ minimal such that $\delta_{n_1} \leq \epsilon$. In the case that $(\mathcal{M}, d)$ is totally ordered with order respecting metric

$$d(\Gamma^1_{n_1}(A), \Xi(A)) \leq d(\Gamma^1_{n_1}(A), \Gamma^2_{n_1}(A)),$$

and we can take n_1 large such that the right hand side is less than the given ϵ (recall we can compute the right hand side to arbitrary precision). Set $\Gamma(A) = \Gamma^1(A)$, then we have $d(\Gamma(A), \Xi(A)) \leq \epsilon$.

For larger k, we use the same idea, but we must be careful to ensure the first $k-1$ limits exist. For the rest of the proof, $\tilde{d}$ will denote an approximation of d to accuracy $1/n_1$ (which by assumption can always be computed). We first deal with the case $k=2$. Let $\epsilon>0$ and consider the intervals $J_\epsilon^1=[0,\epsilon]$ and $J_\epsilon^2=[2\epsilon,\infty)$. Let $\delta_{n_2,n_1}(A)$ be an approximation of

$$d(\Gamma^1_{n_2,n_1}(A),\Gamma^2_{n_2,n_1}(A))+2\cdot 2^{-n_2}$$

from above to accuracy $1/n_1$. Again note that we can easily construct such approximations. It is clear that $\lim_{n_1\to\infty}\delta_{n_2,n_1}(A)=d(\Gamma^1_{n_2}(A),\Gamma^2_{n_2}(A))+2\cdot 2^{-n_2}=\delta_{n_2}(A)$ and that $d(\Gamma^1_{n_2}(A),\Xi(A))\leq\delta_{n_2}(A)$ (again appealing to the lemma if we are in the case of the Attouch–Wets or Hausdorff topologies). Given n_1,n_2, let $l=l(n_2,n_1)\leq n_1$ be maximal such that $\delta_{n_2,l}(A)\in J_\epsilon^1\cup J_\epsilon^2$. If no such l exists or $\delta_{n_2,l}(A)\in J_\epsilon^1$ then define $\mathrm{Osc}(\epsilon;n_1,n_2,A)=1$, otherwise define $\mathrm{Osc}(\epsilon;n_1,n_2,A)=0$. Since $\delta_{n_2,n_1}(A)$ cannot oscillate infinitely often between the two intervals J_ϵ^1 and J_ϵ^2, it follows that

$$\mathrm{Osc}(\epsilon;n_2,A)=\lim_{n_1\to\infty}\mathrm{Osc}(\epsilon;n_1,n_2,A)$$

exists. Define $\Gamma^\epsilon_{n_1}(A)$ as follows. Choose $j\leq n_1$ minimal such that $\mathrm{Osc}(\epsilon;n_1,j,A)=1$ if such a j exists, and define $\Gamma^\epsilon_{n_1}(A)=\Gamma^1_{j,n_1}(A)$. If no such j exists then define $\Gamma^\epsilon_{n_1}(A)=C_0$ where C_0 is a fixed member of $(\mathcal{M},d)$. In particular, $\Gamma^\epsilon_{n_1}$ is a type α algorithm. Now for large n_2, we must have $\delta_{n_2}(A)<\epsilon$ and hence $\mathrm{Osc}(\epsilon;n_2,A)=1$. It follows that $\Gamma^\epsilon(A)=\lim_{n_1\to\infty}\Gamma^\epsilon_{n_1}(A)$ exists and is equal to $\Gamma^1_N(A)$ where $N\in\mathbb{N}$ is minimal with $\mathrm{Osc}(\epsilon;N,A)=1$. It follows that $d(\Gamma^\epsilon(A),\Xi(A))\leq 2\epsilon$.

We use the $\Gamma^\epsilon_{n_1}(A)$ to construct a height-one tower. Observe first that by our assumptions we can compute $\tilde{d}(\Gamma^{\epsilon_1}_m(A),\Gamma^{\epsilon_2}_n(A))$ for $m,n\in\mathbb{N}$ and $\epsilon_1,\epsilon_2>0$. Given n_1, choose $j=j(n_1)\leq n_1$ maximal such that for all $1\leq l\leq j$,

$$\tilde{d}(\Gamma^{2^{-j}}_{n_1}(A),\Gamma^{2^{-l}}_{n_1}(A))\leq 4(2^{-j}+2^{-l}).\tag{2}$$

If no such j exists then set $\Gamma_{n_1}(A)=C_0$, otherwise set $\Gamma_{n_1}(A)=\Gamma^{2^{-j(n_1)}}_{n_1}(A)$. Again, this is easily seen to be a type α algorithm. Pick $N\in\mathbb{N}$, then by the convergence of the $\Gamma^\epsilon_{n_1}(A)$ and $d(\Gamma^\epsilon(A),\Xi(A))\leq 2\epsilon$, (2) must hold for $j=N$ and $1\leq l\leq N$ if n_1 is large enough. Hence by definition of $j(n_1)$,

$$\limsup_{n_1\to\infty} d(\Gamma_{n_1}(A),\Xi(A))\leq\limsup_{n_1\to\infty} d(\Gamma^{2^{-N}}_{n_1}(A),\Xi(A))+2^{3-N}\leq 2^{4-N}.$$

Since N was arbitrary, we must have convergence to $\Xi(A)$.

We now deal with $k=3$. The strategy will be similar to the $k=2$ case but now we construct $\Gamma^\epsilon_{n_2,n_1}(A)$ such that $\Gamma^\epsilon_{n_2}(A)=\lim_{n_1\to\infty}\Gamma^\epsilon_{n_2,n_1}(A)$ exists and is 3ϵ close to $\Xi(A)$ for large n_2, but may not converge in $(\mathcal{M},d)$. Using this, we will construct a height two type α tower. Let $\epsilon>0$ and consider the intervals $J_\epsilon^1=[0,\epsilon]$ and $J_\epsilon^2=[2\epsilon,\infty)$. Let $\delta_{n_3,n_2,n_1}(A)$ be an approximation of

$$d(\Gamma^1_{n_3,n_2,n_1}(A),\Gamma^2_{n_3,n_2,n_1}(A))+2\cdot 2^{-n_3},$$

from above to accuracy $1/n_1$. Again, we have that

$$\lim_{n_2\to\infty}\lim_{n_1\to\infty}\delta_{n_3,n_2,n_1}(A)=d(\Gamma^1_{n_3}(A),\Gamma^2_{n_3}(A))+2\cdot 2^{-n_3}=\delta_{n_3}(A)$$

exists with $d(\Gamma^1_{n_3}(A),\Xi(A))\leq\delta_{n_3}(A)$. Given n_1,n_2 and j, let $l(j,n_2,n_1)\leq n_1$ be maximal such that $\delta_{j,n_2,l}(A)\in J_\epsilon^1\cup J_\epsilon^2$. If no such l exists or $\delta_{j,n_2,l}(A)\in J_\epsilon^1$ then define $\mathrm{Osc}(\epsilon;n_1,n_2,j,A)=1$, otherwise define $\mathrm{Osc}(\epsilon;n_1,n_2,j,A)=0$. Arguing as before, we see that the following limit exists:

$$\mathrm{Osc}(\epsilon;n_2,j,A)=\lim_{n_1\to\infty}\mathrm{Osc}(\epsilon;n_1,n_2,j,A).$$

Now consider $\mathrm{Osc}(\epsilon;n_1,n_2,j,A)$ for $j\leq n_2$. If such a j exists with $\mathrm{Osc}(\epsilon;n_1,n_2,j,A)=1$ then let $j(n_1,n_2)$ be the minimal such j and set $\Gamma^\epsilon_{n_2,n_1}(A)=\Gamma^1_{j(n_1,n_2),n_2,n_1}(A)$. Otherwise set $\Gamma^\epsilon_{n_2,n_1}(A)=C_0$, where again C_0 is some fixed member of $(\mathcal{M},d)$. Since we only deal with finitely many $j\leq n_2$, it is clear that $\Gamma^\epsilon_{n_2,n_1}$ is a type α algorithm. Furthermore, we must have that $\Gamma^\epsilon_{n_2}(A)=\lim_{n_1\to\infty}\Gamma^\epsilon_{n_2,n_1}(A)$ exists and is defined as follows. Let $j(n_2)\leq n_2$ be minimal with $\mathrm{Osc}(\epsilon;n_2,j,A)=1$ (if such a j exists). If such a j exists then $\Gamma^\epsilon_{n_2}(A)=\Gamma^1_{j(n_2),n_2}(A)$, otherwise $\Gamma^\epsilon_{n_2}(A)=C_0$.

Now there exists $N\in\mathbb{N}$ such that $\delta_N(A)<\epsilon/2$ and hence $\delta_{N,n_2}(A)<\epsilon$ for large n_2. But this implies that $\mathrm{Osc}(\epsilon;n_2,N,A)=1$. Hence for n_2 large we must have $j(n_2)\leq N$. If $\delta_l(A)>2\epsilon$ then for large n_2 we must have $\delta_{l,n_2}(A)>2\epsilon$ and hence $\mathrm{Osc}(\epsilon;n_2,l,A)=0$. As n_2 increases, $j(n_2)$ may not converge. However, the above arguments show that for large n_2 it can take only finitely many values, say in the set $S = \{s_1, ..., s_m\}$, all of which must have $\delta_{s_i}(A) \leq 2\epsilon$. It follows that for large n_2 we must have

$$d(\Gamma^{\epsilon}_{n_2}(A), \Xi(A)) \leq 3\epsilon. \tag{3}$$

Now we get to work using these 'towers' (which do not necessarily converge in the last limit) and the trick to avoid oscillations. Define

$$F(n_1, n_2, j, l, A) = \tilde{d}(\Gamma^{2^{-j}}_{n_2,n_1}(A), \Gamma^{2^{-l}}_{n_2,n_1}(A)), \quad F(n_2, j, l, A) = \lim_{n_1\to\infty} F(n_1, n_2, j, l, A) = d(\Gamma^{2^{-j}}_{n_2}(A), \Gamma^{2^{-l}}_{n_2}(A))$$

and the intervals $J^1_{j,l} = [0, 4(2^{-j} + 2^{-l})]$, $J^2_{j,l} = [8(2^{-j} + 2^{-l}), \infty)$. Given j, l, n_1 and n_2, we define $i(j, l, n_2, n_1) \leq n_1$ to be maximal such that $F(i, n_2, j, l, A) \in J^1_{j,l} \cup J^2_{j,l}$. If no such i exists or if it does and $F(i, n_2, j, l, A) \in J^1_{j,l}$ then define $\widehat{\mathrm{Osc}}(n_1, n_2, j, l, A) = 1$, otherwise define $\widehat{\mathrm{Osc}}(n_1, n_2, j, l, A) = 0$. Choose $j = j(n_1, n_2) \leq n_2$ maximal such that for all $1 \leq l \leq j$ we have $\widehat{\mathrm{Osc}}(n_1, n_2, j, l, A) = 1$. If no such j exists then set $\Gamma_{n_2,n_1} = C_0$, otherwise set $\Gamma_{n_2,n_1}(A) = \Gamma^{2^{-j(n_1,n_2)}}_{n_2,n_1}(A)$. Again, this is easily seen to be a type α algorithm.

Arguing as before, we have the existence of

$$\widehat{\mathrm{Osc}}(n_2, j, l, A) = \lim_{n_1\to\infty} \widehat{\mathrm{Osc}}(n_1, n_2, j, l, A).$$

Now define $h = h(n_2) \leq n_2$ maximal such that for all $1 \leq l \leq h$ we have $\widehat{\mathrm{Osc}}(n_2, h, l, A) = 1$. If no such h exists then we must have

$$\Gamma_{n_2}(A) = \lim_{n_1\to\infty} \Gamma_{n_2,n_1}(A) = C_0,$$

otherwise we must have

$$\Gamma_{n_2}(A) = \lim_{n_1\to\infty} \Gamma_{n_2,n_1}(A) = \Gamma^{2^{-h(n_2)}}_{n_2}(A).$$

By (3), for each fixed j, l we have $\widehat{\mathrm{Osc}}(n_2, j, l, A) = 1$ for large n_2 and hence $h(n_2)$ exists for large n_2 and diverges to ∞. For every $N \in \mathbb{N}$,

$$\begin{aligned}\limsup_{n_2\to\infty} d(\Gamma^{2^{-h(n_2)}}_{n_2}(A), \Xi(A)) &\leq \limsup_{n_2\to\infty} \left[d\left(\Gamma^{2^{-N}}_{n_2}(A), \Xi(A)\right) + d\left(\Gamma^{2^{-h(n_2)}}_{n_2}(A), \Gamma^{2^{-N}}_{n_2}(A)\right)\right] \\ &\leq 3 \cdot 2^{-N} + \limsup_{n_2\to\infty} 8(2^{-h(n_2)} + 2^{-N}) \leq 11 \cdot 2^{-N}.\end{aligned}$$

Since N was arbitrary, we must have convergence to $\Xi(A)$. □

Proof of Theorem 2.2.11. The statements regarding intersections in (i) and (ii) follow directly from Proposition 2.1 and the following remark – no assumptions regarding the ability to compute distances between outputs of algorithms is necessary when considering general towers. For the sharpness result in (i), we deal with $X = \Sigma$ and the case of $X = \Pi$ follows from an identical argument. Suppose that $\Delta^G_k \not\ni \{\Xi, \Omega\} \in \Sigma^\alpha_k$. If $\{\Xi, \Omega\} \in \Pi^\alpha_k$, we would have $\{\Xi, \Omega\} \in \Sigma^\alpha_k \cap \Pi^\alpha_k \subset \Sigma^G_k \cap \Pi^G_k = \Delta^G_k$, a contradiction. □

Exercise 2.11

Here is one way of proving the result. This proof is adapted from J. Ben-Artzi, M. Colbrook, A. Hansen, O. Nevanlinna, M. Seidel, *"Computing Spectra - On the solvability complexity index hierarchy and towers of algorithms"*.

Definition 2.2 (Initial segment)**.** *We call a finite matrix $\sigma \in \mathbb{C}^{n\times m}$ an initial segment for an infinite matrix $A \in \Omega_2$ and say that A is an extension of σ if σ is in the upper left corner of A. In particular, $\sigma = \mathcal{P}_n A \mathcal{P}^*_m$ for some $n, m \in \mathbb{N}$, where $\mathcal{P}_n$ is as usual the orthogonal projection onto* $\mathrm{span}\{e_j\}^n_{j=1}$*, where $\{e_j\}_{j\in\mathbb{N}}$ is the canonical basis for $\ell^2(\mathbb{N})$.*

The set $E(\sigma)$ of all extensions of σ is a non-empty open and closed neighbourhood of every extension of σ.

Lemma 2.3. *Let $\{\Gamma_n\}_{n\in\mathbb{N}}$ be a sequence of general algorithms mapping $\Omega_2 \to \mathcal{M}_{\mathrm{dec}}$, $T \subset \Omega_2$ be a non-empty closed set, and $S \subset T$ be a non-meagre set (in T) such that $\xi = \lim_{n\to\infty} \Gamma_n(A)$ exists and is the same for all $A \in S$. Then there exists an initial segment σ and a number n_0 such that $E^T(\sigma) = T \cap E(\sigma)$ is not empty, and such that $\Gamma_n(A) = \xi$ for all $A \in E^T(\sigma)$ and all $n \geq n_0$.*

Proof. We are in a complete metric space T and

$$S = \bigcup_{k\in\mathbb{N}} S_k \quad \text{with} \quad S_k = \{A \in S : \Gamma_n(A) = \xi \ \forall n \geq k\}.$$

Since S is non-meagre, not all of the S_k can be meagre, hence there is a non-meagre S_k, and we set $n_0 = k$.

Now, let A be in the closure $\overline{S_{n_0}}$, i.e., there is a sequence $\{A_j\} \subset S_{n_0}$ converging to A. By the definition of a general algorithm, we have that, for every fixed $n \geq n_0$, $|\Lambda_{\Gamma_n}(A)| < \infty$. That is, Γ_n only depends on a finite part of A, in particular $\{A_f\}_{f\in\Lambda_{\Gamma_n}(A)}$ where $A_f = f(A)$. Since each $f \in \Lambda_{\Gamma_n}(A)$ represents a coordinate evaluation of A and by the definition of the metric d_{B}, it follows that for all sufficiently large j, $f(A) = f(A_j)$ for all $f \in \Lambda_{\Gamma_n}(A)$. By the definition of a general algorithm, it then follows that $\Lambda_{\Gamma_n}(A_j) = \Lambda_{\Gamma_n}(A)$ and $\Gamma_n(A) = \Gamma_n(A_j) = \xi$ for all sufficiently large j. Thus, $\Gamma_n(A) = \xi$ for all $n \geq n_0$ and all $A \in \overline{S_{n_0}}$. Since S_{n_0} is not nowhere dense, we can choose a point $\tilde{A}$ in the interior of $\overline{S_{n_0}}$ and fix a sufficiently large initial segment σ of $\tilde{A}$ such that $E^T(\sigma)$ is a subset of $\overline{S_{n_0}}$. The lemma now follows. □

Roughly speaking, this shows that there is a nice open and closed non-meagre subset of T for which $\lim_{n\to\infty}\Gamma_n(A)$ exists even in a uniform manner. Suppose for a contradiction that $\{\Xi_{2,Q}, \Omega_2, \mathcal{M}_{\mathrm{dec}}, \Lambda_2\} \in \Delta_3^G$ with height-two tower of general algorithms $\{\Gamma_{r,s}\}$. Let Γ_r denote, as usual, the pointwise limits $\lim_{s\to\infty}\Gamma_{r,s}$. We will inductively construct initial segments $\{\sigma_n\}$ with $\sigma_{n+1} \supset \sigma_n$ yielding an infinite matrix $A \supset \sigma_n$ for all $n \in \mathbb{N}$, such that $\lim_{r\to\infty}\Gamma_r(A)$ does not exist. We construct $\{\sigma_n\}$ with the help of two sequences of subsets $\{T_n\}$ and $\{S_n\}$ of Ω_2, with the properties that $T_{n+1} \subset T_n$, each T_n is closed, and either $T_n = \Omega_2$ or there is an initial segment $\sigma \in \mathbb{C}^{m\times m}$ where $m \geq n$ such that T_n is the set of all extensions of σ with all the remaining entries in the first n columns being zero.

We construct the sets T_n and S_n inductively. Let $T_0 = \Omega_2$. Suppose that we have chosen T_n. The subset of all matrices in T_n with one particular entry being fixed is closed in T_n. Hence, the set of all matrices with one particular column being fixed is closed (as an intersection of closed sets). The latter set has no interior points in T_n; hence, it is nowhere dense in T_n. This provides that the set of all matrices in T_n for which a particular column has only finitely many 1s is a countable union of nowhere dense sets in T_n, hence is meagre in T_n. Hence, the set of all matrices in $A \in T_n$ with $\Xi_{2,Q}(A) = 0$ (i.e., matrices with infinitely many 'finite columns') is meagre in T_n as well. Let R be its complement in T_n, i.e., the non-meagre set of all matrices $A \in T_n$ with $\Xi_{2,Q}(A) = 1$.

Clearly, $R = \cup_{r\in\mathbb{N}} R_r$ with $R_r = \{A \in R : \Gamma_k(A) = 1 \ \forall k \geq r\}$, and there is an r_n such that $S_n = R_{r_n}$ is non-meagre in T_n. Note that $\Gamma_{r_n,s}$ are general algorithms and $\Gamma_{r_n}(A) = \lim_{s\to\infty}\Gamma_{r_n,s}(A) = 1$ for all $A \in S_n$. Thus, Lemma 2.3 applies and yields an initial segment σ_n, such that

$$E^{T_n}(\sigma_n) \neq \emptyset \quad \text{and } \Gamma_{r_n}(A) = 1 \text{ for all } A \in E^{T_n}(\sigma_n). \tag{4}$$

Now, let $T_{n+1} \subset T_n$ be the (closed) set of all matrices in $E^{T_n}(\sigma_n)$ with all remaining (i.e., outside the initial segment σ_n) entries in the first $n+1$ columns being zero. This completes the construction.

The nested initial segments $\sigma_{n+1} \supset \sigma_n$ yield a matrix $A \in \cap_{n=0}^{\infty} T_n$ and this A has only finitely many 1s in each of its columns. Thus, $\Xi_{2,Q}(A) = 0$. However, by the construction of $\{T_n\}$, we have that $A \in E^{T_n}(\sigma_n)$ for all $n \in \mathbb{N}$. By (4), $\Gamma_k(A) = 1$ for infinitely many k, a contradiction.

Exercise 2.12

Proof of propositions: We first prove the first proposition. Fix $A \in \Omega$ throughout the proof and let $H(a) = \Gamma(A, a)$. Suppose that $\{a_n\} \subset C$ with $\lim_{n\to\infty} a_n = a \in C$, where $\Lambda_\Gamma(A, a)$ is finite. Since $\Lambda_\Gamma(A, a)$ is finite, there exists an N such that if $n \geq N$, then $p_j(a_n) = p_j(a)$ for all $p_j \in \Lambda_\Gamma(A, a)$. It follows from the definition of a probabilistic general algorithm that $\Lambda_\Gamma(A, a_n) = \Lambda_\Gamma(A, a)$ and $H(a_n) = H(a)$. Since the set $\{a \in C : |\Lambda_\Gamma(A, a)| < \infty\}$ and $\mathcal{M} \cup \mathrm{NH}$ are metrisable, this shows that H is continuous on $\{a \in C : |\Lambda_\Gamma(A, a)| < \infty\}$.

To prove measurability, it is enough to show that the preimage $H^{-1}(U)$ of each open subset $U \subset \mathcal{M} \cup \{\mathrm{NH}\}$ is a Borel subset of C. If $\mathrm{NH} \notin U$, this follows from the continuity of H on $\{a \in C : |\Lambda_\Gamma(A, a)| < \infty\}$. Hence, it is enough to show that $H^{-1}(\{\mathrm{NH}\})$ is Borel subset. This preimage can be written as the union

$$\{a \in C : |\Lambda_\Gamma(A, a)| = \infty\} \cup \{a \in C : |\Lambda_\Gamma(A, a)| < \infty, H(a) = \mathrm{NH}\}.$$

Using the above arguments, the first set in this union is closed, as its complement is open. Similarly, the second set is open. Hence, the result follows.

To prove the second proposition, suppose that $B \in \Omega$ is such that $f(A) = f(B)$ for all $f \in S \cap \Lambda$. If $a \in C$ with $\Lambda_\Gamma(A, a) \subset S$, then we must have $\Gamma(A, a) = \Gamma(B, a)$ and $\Lambda_\Gamma(A, a) = \Lambda_\Gamma(B, a)$. It follows that

$$\{a \in C : \Lambda_\Gamma(A, a) \subset S\} \subset \{a \in C : \Lambda_\Gamma(B, a) \subset S\}.$$

We can reverse the roles of A and B to see that

$$\{a \in C : \Lambda_\Gamma(B, a) \subset S\} \subset \{a \in C : \Lambda_\Gamma(A, a) \subset S\}.$$

We also clearly have $\Lambda_\Gamma(A, a) = \Lambda_\Gamma(B, a)$ on this set.

Equivalence of $\Delta_1^{\mathbb{P}}$ and Δ_1^G when Λ is countable: It is clear that if a problem lies in Δ_1^G, then it lies in $\Delta_1^{\mathbb{P}}$. For the converse, suppose that $\{\Xi, \Omega, \mathcal{M}, \Lambda\} \in \Delta_1^{\mathbb{P}}$. Let $n \in \mathbb{N}$, then there exists a probabilistic general algorithm Γ' such that

$$\mathbb{P}\left(\left\{a \in C : d(\Gamma'(A, a), \Xi(A)) \leq 2^{-(n+1)}\right\}\right) > 2/3 \quad \forall A \in \Omega. \tag{5}$$

Since the set of finite subsets of $\Lambda^{\mathbb{P}}$ is countable, there exists an increasing sequence of finite sets $S_1 \subset S_2 \subset S_3 \subset \cdots \subset \Lambda^{\mathbb{P}}$ such that if $S \subset \Lambda^{\mathbb{P}}$ is finite, then $S \subset S_M$ for sufficiently large M. Let $A \in \Omega$ be an input, for a finite set $S \subset \Lambda^{\mathbb{P}}$, define the set

$$U_S = \{a \in C : \Lambda_{\Gamma'}(A, a) \subset S\}.$$

We claim that U_S is open and hence measurable. To see this, suppose that $a \in U_S$. Since $\Lambda_{\Gamma'}(A, a) \subset S$ is finite, the definition of a probabilistic general algorithm implies that there exists $N \in \mathbb{N}$ such that if $b \in C$ with $p_n(b) = p_n(a)$ for all $n \leq N$, then $\Lambda_{\Gamma'}(A, b) = \Lambda_{\Gamma'}(A, a)$ and $\Gamma'(A, b) = \Gamma'(A, a)$. In particular, $b \in U_S$. Hence, an open neighborhood of a exists inside U_S, and the claim follows.

If $a \in C$ has $\Gamma'(A, a) \neq \mathrm{NH}$, then $\Lambda_{\Gamma'}(A, a)$ is finite. It follows that

$$\{a \in C : \Gamma'(A, a) \neq \mathrm{NH}\} \subset \bigcup_{S \subset \Lambda^{\mathbb{P}}, S \text{ finite}} U_S =: U,$$

where U is open and hence measurable. Then, by (5) and that NH is an isolated point, it follows that $\mathbb{P}(U) > 2/3$. Moreover, the partition of $\Lambda^{\mathbb{P}}$ defined above implies that $U = \bigcup_{m=1}^{\infty} U_{S_m}$. Since the sets $\{U_{S_m}\}_m$ are increasing and $\mathbb{P}(U) > 2/3$, we may choose m minimal such that $\mathbb{P}(U_{S_m}) > 2/3$. We claim that with the choice $S(A) = S_m$, the map $A \mapsto S(A)$ is a general algorithm with $\Lambda_S(A) = \Lambda \cap S(A)$. To see this, suppose that $B \in \Omega$ with $f(A) = f(B)$ for all $f \in \Lambda \cap S(A)$. The proposition proven above implies that

$$\mathbb{P}(\{a \in C : \Lambda_{\Gamma'}(B, a) \subset S(A)\}) = \mathbb{P}(\{a \in C : \Lambda_{\Gamma'}(A, a) \subset S(A)\}) > 2/3.$$

Since we chose the set S_m with m minimal in the above construction, it follows that $S(B) \subset S(A)$. Since $f(A) = f(B)$ for all $f \in \Lambda \cap S(B) \subset \Lambda \cap S(A)$, we apply the proposition again (with the roles of A and B reversed) to see that

$$\mathbb{P}(\{a \in C : \Lambda_{\Gamma'}(A, a) \subset S(B)\}) = \mathbb{P}(\{a \in C : \Lambda_{\Gamma'}(B, a) \subset S(B)\}) > 2/3.$$

Again, using the minimality of m, we see that $S(A) \subset S(B)$. It follows that $S(A) = S(B)$ and $\Lambda_S(A) = \Lambda_S(B)$. Hence, S is a general algorithm as claimed.

Given $A \in \Omega$, let

$$E(A) = \left\{a \in C : d(\Gamma'(A, a), \Xi(A)) \leq 2^{-(n+1)}\right\},$$

which is measurable by the same arguments as for U_S. Since $\mathbb{P}(E(A)) > 2/3$ and $\mathbb{P}(U_{S(A)}) > 2/3$, it follows immediately that $\mathbb{P}(E(A) \cap U_{S(A)}) > 1/3$. Let $\Gamma(A)$ be a member of $\mathcal{M}$ that satisfies

$$\mathbb{P}\left(U_{S(A)} \cap \left\{a \in C : d(\Gamma(A), \Gamma'(A, a)) \leq 2^{-(n+1)}\right\}\right) > 1/3.$$

Such a choice $\Gamma(A)$ must always exist (since the above is satisfied by $\Xi(A)$ since $\mathbb{P}(E(A) \cap U_{S(A)}) > 1/3$).

We claim that $\Gamma(A)$ is a general algorithm with $\Lambda_\Gamma(A) = \Lambda \cap S(A)$. We first compute $S(A)$ using $\Lambda_S(A) = \Lambda \cap S(A)$. Let N be such that for all $n > N$, $p_n \notin S(A)$. Note that if $a, a' \in C$ is such that $p_j(a) = p_j(a')$ for $j = 1, \ldots, N$, then $\Lambda_{\Gamma'}(A, a) \subset S(A)$ if and only if $\Lambda_{\Gamma'}(A, a') \subset S(A)$. Indeed, without loss of generality suppose that $\Lambda_{\Gamma'}(A, a) \subset S(A)$; then for every $f \in \Lambda_\Gamma(A, a)$, $f(A, a) = f(A, a')$ so by the definition of a probabilistic general algorithm, $\Lambda_\Gamma(A, a) =$

$\Lambda_\Gamma(A, a')$. In particular, we can construct $U_{S(A)}$ using only the first N projections. Then for each $a \in U_{S(A)}$, we may compute $\Gamma'(A, a)$ using only the finitely many evaluation functions in $\Lambda \cap S(A)$. Next, we must find $\Gamma(A)$ such that $d(\Gamma(A), \Gamma'(A, a)) \le 2^{-(n+1)}$ for at least $(2/3)/\mathbb{P}(U_{S(A)})$ of the $a \in U_{S(A)}$; such a $\Gamma(A)$ exists (as $\Xi(A)$ is one) and can be computed deterministically, using only the evaluation functions in $\Lambda \cap S(A)$, by simulating all possible combinations of N coin tosses. Finally, suppose $A, B \in \Omega$ are such that $f(A) = f(B)$ for all $f \in \Lambda \cap S(A)$. Then as $S(A) = S(B)$, by the proposition proven above, $\{a \in C : \Lambda_{\Gamma'}(A, a) \subset S(A)\} = \{a \in C : \Lambda_{\Gamma'}(B, a) \subset S(B)\}$ and so for all $a \in U_{S(A)}$, $\Gamma'(A, a) = \Gamma'(B, a)$. Hence, since $\Gamma(A)$ and $\Gamma(B)$ are chosen according to the same conditions, we can choose $\Gamma(A)$ consistently to make it a general algorithm. Note then that

$$\mathbb{P}\left(E(A) \cap U_{S(A)} \cap \left\{a \in C : d(\Gamma(A), \Gamma'(A, a)) \le 2^{-(n+1)}\right\}\right) > 2/3 + 1/3 - 1 = 0.$$

Hence, there exists a in this intersection so that

$$d(\Gamma(A), \Xi(A)) \le d(\Gamma(A), \Gamma'(A, a)) + d(\Gamma'(A, a), \Xi(A)) \le 2^{-n}.$$

Since $n \in \mathbb{N}$ was arbitrary, it follows that $\{\Xi, \Omega, \mathcal{M}, \Lambda\} \in \Delta_1^G$.

Equivalence of $\Delta_2^{\mathbb{P}}$ and Δ_2^G when Λ is countable: It is clear that if a problem lies in Δ_2^G, then it lies in $\Delta_2^{\mathbb{P}}$. For the converse, suppose that $\{\Xi, \Omega, \mathcal{M}, \Lambda\} \in \Delta_2^{\mathbb{P}}$. Then there exists an SPGA $\{\Gamma'_n\}$ such that

$$\mathbb{P}\left(\left\{a \in C : \lim_{m\to\infty} \Gamma'_m(A, a) = \Xi(A)\right\}\right) > 2/3 \text{ and } \mathbb{P}\left(\{a \in C : \Gamma'_n(A, a) \neq \mathrm{NH}\}\right) > 2/3 \quad \forall n \in \mathbb{N}, A \in \Omega.$$

Let $N \in \mathbb{N}$. Arguing as above, there exist general algorithms (now dependent on n) $\{S_n\}$ such that each $S_n(A)$ is a finite subset of $\Lambda^{\mathbb{P}}$ that satisfy

$$\mathbb{P}\left(\left\{a \in C : \Lambda_{\Gamma'_n}(A, a) \subset S_n(A)\right\}\right) > 2/3.$$

We now consider the set

$$R_n(A) = \left\{x \in \mathcal{M} : \mathbb{P}\left(U_{S_n(A)} \cap \left\{a \in C : d(\Gamma'_n(A, a), x) \le 2^{-N}\right\}\right) > 1/3\right\}.$$

Let x_0 be a fixed member of $\mathcal{M}$. If $R_n(A) = \emptyset$, we set $\Gamma_n^N(A) = x_0$. Otherwise, let $\Gamma_n^N(A) = x$ for some element x of $R_n(A)$. As before, this choice can be executed so that Γ_n^N is a general algorithm.

We have

$$\left\{a \in C : \lim_{m\to\infty} \Gamma'_m(A, a) = \Xi(A)\right\} \subset \bigcup_{M=1}^{\infty} \left\{a \in C : d(\Gamma'_m(A, a), \Xi(A)) \le 2^{-N} \text{ for all } m \ge M\right\}.$$

It follows that there exists some M so that the set

$$E_M(A) = \left\{a \in C : d(\Gamma'_m(A, a), \Xi(A)) \le 2^{-N} \text{ for all } m \ge M\right\}$$

has $\mathbb{P}(E_M(A)) > 2/3$. If $n \ge M$, then $\mathbb{P}(E_M(A) \cap U_{S_n(A)}) > 1/3$. In particular, $\Xi(A) \in R_n(A)$ so that $R_n(A) \neq \emptyset$. Moreover,

$$\mathbb{P}\left(E_M(A) \cap U_{S_n(A)} \cap \left\{a \in C : d(\Gamma'_n(A, a), \Gamma_n^N(A)) \le 2^{-N}\right\}\right) > 2/3 + 1/3 - 1 = 0.$$

If a lies in this intersection, then

$$d(\Gamma_n^N(A), \Xi(A)) \le d(\Gamma_n^N(A), \Gamma'_n(A, a)) + d(\Gamma'_n(A, a), \Xi(A)) \le 2^{1-N}.$$

It follows that

$$\limsup_{n\to\infty} d(\Gamma_n^N(A), \Xi(A)) \le 2^{1-N}. \tag{6}$$

We must now alter $\{\Gamma_n^N\}$ to obtain a Δ_2^G-tower.

Given m, choose $N = N(m) \le m$ maximal such that for all $1 \le l \le N$,

$$d(\Gamma_m^N(A), \Gamma_m^l(A)) \le 4(2^{-N} + 2^{-l}).$$

If no such N exists then set $N(m) = 0$ and $\Gamma_m(A) = x_0$, otherwise set $\Gamma_m(A) = \Gamma_m^{N(m)}(A)$. The bound in (6) implies that $\lim_{m\to\infty} N(m) = \infty$. Furthermore, for each fixed $q \in \mathbb{N}$, since for sufficiently large m $N(m) \ge q$, it follows that

$$\begin{aligned}\limsup_{m\to\infty} d(\Gamma_m(A), \Xi(A)) &\le \limsup_{m\to\infty} d(\Gamma_m^{N(m)}(A), \Gamma_m^q(A)) + \limsup_{m\to\infty} d(\Gamma_m^q(A), \Xi(A)) \\ &\le \limsup_{m\to\infty} 4(2^{-N(m)} + 2^{-q}) + 2^{1-q} = 6 \cdot 2^{-q}.\end{aligned}$$

Since q was arbitrary, we have the desired convergence. It follows that $\{\Xi, \Omega, \mathcal{M}, \Lambda\} \in \Delta_2^G$.

3 Chapter 3

Exercise 3.1

We have

$$(J_k - zI)^{-1} = \begin{pmatrix} \frac{-1}{z} & \frac{-1}{z^2} & \cdots & \frac{-1}{z^k} \\ & \ddots & \ddots & \vdots \\ & & \frac{-1}{z} & \frac{-1}{z^2} \\ & & & \frac{-1}{z} \end{pmatrix}.$$

Let $u = (u_1, \dots, u_k) \in \mathbb{C}^k$ with $\|u\| \leq 1$, then

$$\left|[(J_k - zI)^{-1}u]_j\right| = \left|\sum_{l=1}^{k+1-j} \frac{-u_{l+j-1}}{z^l}\right| \leq \sum_{l=1}^{k+1-j} \frac{1}{|z|^l}.$$

It follows that

$$\|(J_k - zI)^{-1}\|^2 \leq \sum_{j=1}^{k}\left[\sum_{l=1}^{k+1-j}\frac{1}{|z|^l}\right]^2 = \sum_{j=1}^{k}\left[\sum_{l=1}^{j}\frac{1}{|z|^l}\right]^2 = \sum_{j=1}^{k}\left(\frac{1-(1/|z|)^j}{1-|z|}\right)^2 \leq \frac{1}{(1-|z|)^2}\sum_{j=1}^{k}\frac{1}{|z|^{2j}} = O(|z|^{-2k}).$$

Combining with the lower bound from the book, we see that $\|(J_k - zI)^{-1}\| \asymp |z|^{-k}$.

If $|z| = 1$, we can bound $\|(J_k - zI)^{-1}\|$ by the Frobenius norm to see that $\|(J_k - zI)^{-1}\| = O(k)$. Let $u = (-z, -z^2, \dots, -z^k)$, then

$$(J_k - zI)^{-1}u = (k, (k-1)z, \dots, z^{k-1}).$$

Hence,

$$\frac{\|(J_k - zI)^{-1}u\|}{\|u\|} = \frac{\sqrt{1 + 2^2 + \ldots + k^2}}{\sqrt{k}} \geq \frac{k}{\sqrt{3}}.$$

It follows that $\|(J_k - zI)^{-1}\| \asymp k$.

Exercise 3.2

First suppose that f is a smooth compactly supported function on $\mathbb{R}$. Using integration by parts, Hölder's inequality and the AM–GM inequality, we have that

$$\|f'\|^2 \leq \|f\|\|f''\| \leq \frac{1}{2}\left(a\|f''\|^2 + a^{-1}\|f\|^2\right)$$

for every $a > 0$. Using integration by parts once more, we have

$$\begin{aligned}\|Tf\|^2 = \|f''\|^2 + \|xf\|^2 + 2\mathrm{Re}(\langle \mathrm{i}xf, -f''\rangle) &= \|f''\|^2 + \|xf\|^2 + 2\mathrm{Re}(\langle \mathrm{i}f, f'\rangle) \\ &\geq \|f''\|^2 + \|xf\|^2 - 2\|f\|\|f'\| \geq \|f''\|^2 + \|xf\|^2 - a\|f'\|^2 - a^{-1}\|f\|^2.\end{aligned}$$

Combining these inequalities for sufficiently small a, we see that there exists a constant C with

$$\|f''\|^2 + \|xf\|^2 \leq C\left(\|Tf\|^2 + \|f\|^2\right). \tag{7}$$

Using the density of such f in $\mathcal{D}(T)$ and the fact that T is closed, we arrive at

$$\mathcal{D}(T) = \{f \in H^2(\mathbb{R}) : xf \in L^2(\mathbb{R})\}.$$

We would like to use Rellich's criterion to conclude that T has compact resolvent. First, let $\lambda \in \mathbb{R}_{<0}$ and f be a smooth compactly supported function on $\mathbb{R}$. Then, using the same arguments as above,

$$\begin{aligned}\|(T - \lambda I)f\|^2 &= \|f''\|^2 + \|(ix - \lambda)f\|^2 + 2\mathrm{Re}(\langle(\mathrm{i}x - \lambda)f, -f''\rangle) \\ &= \|f''\|^2 + \|(ix - \lambda)f\|^2 - \lambda\|f'\|^2 + 2\mathrm{Re}(\langle \mathrm{i}xf, f'\rangle) \\ &\geq \|f''\|^2 + \|(ix - \lambda)f\|^2 - \lambda\|f'\|^2 - a\|f'\|^2 - a^{-1}\|f\|^2.\end{aligned}$$

Note that $|ix - \lambda|$ is bounded below on $\mathbb{R}$ (as a function of x). We can now choose a and $|\lambda|$ sufficiently large so that $\|(T - \lambda I)f\|$ is bounded below by a constant factor of $\|f\|$. Applying a density argument, we see that $\sigma_{\inf}(T - \lambda I) > 0$. We argue similarly for the adjoint to see that $\lambda \notin \mathrm{Sp}(T)$. Now let B be the closed unit ball in $L^2(\mathbb{R})$ and $g \in B$. Let $f = (T - \lambda I)^{-1} g$, then

$$\|Tf\| \leq |\lambda|\|f\| + \|g\| \leq |\lambda|\|(T - \lambda I)^{-1}\| + 1, \quad \|f\| \leq \|(T - \lambda I)^{-1}\|.$$

It follows from (7) that the image of B under $(T - \lambda I)^{-1}$ is a subset of

$$\{f \in L^2(\mathbb{R}) : \|f\| \leq C, \|xf\| \leq C, \|f''\| \leq C\}$$

for some $C > 0$. Using Rellich's criterion, we see that $(T - \lambda I)^{-1}B$ is a subset of a compact set and, hence, relatively compact. So $(T - \lambda I)^{-1}$ is compact and, hence, the spectrum of T can only consist of isolated eigenvalues.

Suppose that $f \in \mathcal{D}(T)$ and set $f_t(x) = f(x - t)$. Then

$$[(T - itI)f_t](x) = -f''(x - t) + i(x - t)f(x - t).$$

Hence, $\sigma_{\inf}(T) = \sigma_{\inf}(T - itI)$. We can argue in a similar fashion with the adjoint to see that $\|(T - zI)^{-1}\|^{-1}$ is constant along each vertical line in the complex plane. Hence, since T has discrete spectrum, its spectrum must be empty.

For the final part, we know (from section 3.4 of the book) that the Hermite functions $\{\psi_m : m \in \mathbb{Z}_{\geq 0}\}$ are such that their span forms a core of T and its adjoint. We now consider the basis functions $\psi_m(x) \otimes \psi_n(y)$ of $L^2(\mathbb{R}^2)$ and the operator

$$S_0 = -\frac{d^2}{dx^2} + ix$$

initially defined on $\mathrm{span}\{\psi_m \otimes \psi_n : m, n \in \mathbb{Z}_{\geq 0}\}$. We then let S be the closure of this operator. The corresponding matrix representation of S is a direct sum of infinitely many copies of that of T. The operator S has empty spectrum, but its resolvent cannot be compact since it is a countable direct sum of a single, nonzero, operator.

Exercise 3.3

It suffices to consider $m = 1$ since the following arguments easily extend. Define $x_0 = \sup_{|z| \leq 1} \mathrm{dist}(z, \mathrm{Sp}(A))$ and assume that $x_0 > 0$, else there is nothing to prove. It suffices to define $g_1(x)$ for $x \in \mathbb{R}_{\geq 0}$ with $x \leq x_0$ since we may then extend g_1 to the whole of $\mathbb{R}_{\geq 0}$. Define

$$g_1^*(x) = \inf_{|z| \leq 1, \mathrm{dist}(z, \mathrm{Sp}(A)) \geq x} \|(A - zI)^{-1}\|^{-1} \quad \text{for} \quad 0 \leq x \leq x_0.$$

Note that $g_1^*(x) \leq x$. By construction, if $|z| \leq 1$, then

$$g_1^*(\mathrm{dist}(z, \mathrm{Sp}(A))) = \inf_{|w| \leq 1, \mathrm{dist}(w, \mathrm{Sp}(A)) \geq \mathrm{dist}(z, \mathrm{Sp}(A))} \|(A - wI)^{-1}\|^{-1} \leq \|(A - zI)^{-1}\|^{-1}.$$

Moreover, since the unit ball in $\mathbb{C}$ is compact and $z \mapsto \mathrm{dist}(z, \mathrm{Sp}(A))$ and $z \mapsto \|(A - zI)^{-1}\|^{-1}$ are continuous, we see that g_1^* is continuous. However, it may not be strictly increasing. To fix this, we define the sequences

$$y_0 = g_1^*(x_0), \quad y_n = \frac{1}{2} y_{n-1}, \quad x_n = \min\left\{x : g_1^*(x) \geq \frac{1}{2} y_{n-1}\right\}, \quad n \in \mathbb{N}.$$

Since $g_1^*(x) > 0$ if $x > 0$ and $g_1^*(0) = 0$, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$. We then consider the continuous piecewise affine function g_1 that satisfies $g_1(x_n) = y_{n+1}$ for $n \in \mathbb{N}$ and is affine on the intervals (x_{n+1}, x_n). It can easily be checked that this function satisfies all the desired properties and $g_1(x) \leq g_1^*(x)$.

For the Jordan block example, we take

$$A = \bigoplus_{k=1}^{\infty} (J_k + kI),$$

defined in the obvious manner as an element of $\Omega_{l^2(\mathbb{N})}$. The spectrum of A is $\mathbb{N}$, but since $\lim_{k\to\infty} \|(J_k - \frac{1}{2}I)^{-1}\| = \infty$,

$$\inf\{\|(A - zI)^{-1}\|^{-1} : \mathrm{dist}(z, \mathrm{Sp}(A)) = 1/2\} = 0.$$

Hence, we cannot use a single function $g_m = g_1$ for all $m \in \mathbb{N}$, regardless of the choice of g_1.

Exercise 3.4

The main alteration is to replace the functions $\{\Phi_n\}$ in `CompSpec`. Let $A \in \Omega_g$. Recall the functions

$$\gamma_{n_2,n_1}(z, A) = \min\left\{\sigma_{\inf}(\mathcal{P}_{n_1}(A - zI)\mathcal{P}_{n_2}^*), \sigma_{\inf}(\mathcal{P}_{n_1}(A^* - \bar{z}I)\mathcal{P}_{n_2}^*)\right\}$$

from Chapter 1. We can compute an approximation to $\gamma_{n_2,n_1}(z, A)$ from *above* to within an accuracy of $1/n_2$ in finitely many arithmetic operations and comparisons. Call this approximation function $\zeta_{n_2,n_1}(z, A)$ and we can assume that it takes values in $\frac{1}{2n_2}\mathbb{N}$. As $n_1 \to \infty$, $\gamma_{n_2,n_1}(z, A)$ converges to $\gamma_{n_2}(z, A) = \min\{\sigma_{\inf}((A - zI)\mathcal{P}_{n_2}^*), \sigma_{\inf}((A^* - \bar{z}I)\mathcal{P}_{n_2}^*)\}$ monotonically from below. By taking successive maxima over n_1 and then minima over n_2, we can assume that $\zeta_{n_2,n_1}(z, A)$ is non-decreasing in n_1 and non-increasing in n_2. Since γ_{n_2,n_1} obeys these monotonicity relations, this preserves the error bound of $1/n_2$. It follows that $\zeta_{n_2,n_1}(z, A)$ converges to $\zeta_{n_2}(z, A)$, which takes values in the set $\frac{1}{2n_2}\mathbb{N}$ and, hence, $\zeta_{n_2,n_1}(z, A)$ is eventually constant for large n_1 with $\|(A - zI)^{-1}\|^{-1} \leq \gamma_{n_2}(z, A) \leq \zeta_{n_2}(z, A) \leq \gamma_{n_2}(z, A) + 1/n_2$.

We now let $\hat{\Gamma}_{n_2,n_1}(A)$ be the output of `CompSpec` with $n = n_2$ and $\zeta_{n_2,n_1}(z, A)$ replacing $\Phi_n(z, A)$. The proof of convergence of `CompSpec` implies that

$$\lim_{n_2\to\infty} \lim_{n_1\to\infty} \hat{\Gamma}_{n_2,n_1}(A) = \mathrm{Sp}(A) \quad \forall A \in \Omega_g,$$

and we must have that $\hat{\Gamma}_{n_2,n_1}(A)$ is constant for large n_1. We must adapt this tower to achieve the Σ_2^A classification.

Let $h_{j,m}$ be an approximation of g_j^{-1} from above to accuracy $1/m$, computed in finitely many arithmetic operations and comparisons. We also assume that each $h_{j,m}$ is an increasing function. We first set

$$\Gamma'_{m,n}(A) = \{z \in \hat{\Gamma}_{m,n}(A) : h_{\lceil |z| \rceil_+,m}(\zeta_{m,n}(z, A)) \leq (|z|^2 + 1)^{-1}\}.$$

This does not alter the convergence and we still have

$$\lim_{n_2\to\infty} \lim_{n_1\to\infty} \Gamma'_{n_2,n_1}(A) = \mathrm{Sp}(A) \quad \forall A \in \Omega_g, \tag{8}$$

For each $\Gamma'_{m,n}(A)$, let $S(m, n, A) = \max_{z\in\Gamma'_{m,n}(A)} h_{\lceil |z| \rceil_+,m}(\zeta_{m,n}(z, A))$, where we take the maximum over the empty set to be $+\infty$. Note that $S(m, n, A)$ can be computed using finitely many arithmetic operations and comparisons from the given data. As before, $\Gamma'_{m,n}(A)$ is constant for large n, and, hence, so is $S(m, n, A)$. Let

$$S_m(A) = \lim_{n\to\infty} S(m, n, A) = \max_{z\in\Gamma'_m(A)} h_{\lceil |z| \rceil_+,m}(\zeta_m(z, A)).$$

The convergence in (8) and the local uniform convergence of $\zeta_m(z, A)$, together with the restriction $h_{\lceil |z| \rceil_+,m}(\zeta_{m,n}(z, A)) \leq (|z|^2 + 1)^{-1}$ imply that $\lim_{m\to\infty} S_m(A) = 0$.

For given $m, n \in \mathbb{N}$, if $n < m$ then set $\Gamma_{m,n}(A) = \emptyset$. Otherwise, compute $S(k, n, A)$ for $m \leq k \leq n$. If there exists such a k with $S(k, n, A) \leq 2^{-m}$, then choose a minimal such k and set $\Gamma_{m,n}(A) = \Gamma'_{k,n}(A)$ (which must be non-empty by the definition of $S(m, n, A)$), otherwise set $\Gamma_{m,n}(A) = \emptyset$. Let $N_1(m) \geq m$ be minimal such that $S_{N_1(m)} \leq 2^{-m}$. It follows that we must have $\lim_{n\to\infty} \Gamma_{m,n}(A) =: \Gamma_m(A) = \Gamma'_{N_1(m)}(A)$. We must also have $\lim_{m\to\infty} \Gamma_m(A) = \mathrm{Sp}(A)$. Furthermore,

$$\max_{z\in\Gamma_m(A)} \mathrm{dist}(z, \mathrm{Sp}(A)) \leq \max_{z\in\Gamma_m(A)} g_{\lceil |z| \rceil_+}^{-1}(\gamma(z, A)) \leq \max_{z\in\Gamma_m(A)} h_{\lceil |z| \rceil_+,N_1(m)}(\zeta_m(z, A)) \leq S_{N_1(m)}(A) \leq 2^{-m}$$

so that $\Gamma_m(A) \subset \mathrm{Sp}(A) + B_{2^{-m}}(0)$. We can easily alter this tower by taking subsequences to obtain Σ_2^A convergence in the Attouch–Wets metric space.

The upper bound for pseudospectra is simpler and follows by replacing $\{\Phi_n\}$ in `PseudoSpec` with $\{\zeta_{m,n}\}$. The lower bound follows the same argument that spectra of self-adjoint operators cannot be computed in one limit.

Exercise 3.5

We have

$$\langle (A - zI)e_i, (A - zI)e_j \rangle = \langle Ae_i, Ae_j \rangle - z\langle e_i, Ae_j \rangle - \bar{z}\langle Ae_i, e_j \rangle + |z|^2 \langle e_i, e_j \rangle$$

and similarly for A^*, so the information in Λ' is sufficient to compute $\langle (A - zI)e_i, (A - zI)e_j \rangle$ and $\langle (A - zI)^* e_i, (A - zI)^* e_j \rangle$ to arbitrary accuracy. By operator folding, there exists an approximation $\{\Phi_n\}_{n\in\mathbb{N}}$ with $\Phi_n \downarrow \gamma$ as $n \to \infty$ with local uniform convergence and, hence, we get a Σ_1^A algorithm through `CompSpec`.

Exercise 3.6

Suppose that $A \in \Omega_{l^2(\mathbb{N})}$ is band-dominated. There exists a sequence $\{A_n\}$ of banded operators that satisfy $\lim_{n\to\infty} \|A - A_n\| = 0$. Fix n, then since $\lim_{m\to\infty} f(m) - m = \infty$, $D_{f,m}(A_n) = 0$ for sufficiently large m. For such an m,

$$D_{f,m}(A) \le D_{f,m}(A_n) + \|A - A_n\| = \|A - A_n\|.$$

It follows that

$$\limsup_{m\to\infty} D_{f,m}(A) \le \|A - A_n\|.$$

We now take $n \to \infty$ to see that $A \in \Omega_f$.

Exercise 3.7

Fix $A \in \Omega_{l^2(\mathbb{N})}$. For $n \in \mathbb{N}$, take $N \in \mathbb{N}$ sufficiently large such that

$$\sum_{i=N+1}^{\infty} |\langle Ae_j, e_i\rangle|^2 < 4^{-n} \quad \text{and} \quad \sum_{i=N+1}^{\infty} |\langle A^*e_j, e_i\rangle|^2 < 4^{-n} \quad \text{for } 1 \le j \le n.$$

Set $f(n) = \max\{N, n+1\}$. Then the columns of $(I - \mathcal{P}_{f(n)}^*\mathcal{P}_{f(n)})A\mathcal{P}_n^*$ have norm at most 2^{-n} and similarly for the columns of $(I - \mathcal{P}_{f(n)}^*\mathcal{P}_{f(n)})A^*\mathcal{P}_n^*$. The operator norm of a (possibly infinite) matrix is bounded by its Frobenius norm. Hence, $D_{f,n}(A) \le \sqrt{n}2^{-n} \to 0$ as $n \to \infty$ so that $A \in \Omega_f$.

Exercise 3.8

We already know that $\{\mathrm{Sp}_\epsilon, \Omega_f, \mathcal{M}_{\mathrm{AW}}, \Lambda\} \in \Sigma_1^A$ for all $\epsilon > 0$. Let $\{\Gamma_n^\epsilon\}_{n\in\mathbb{N}}$ be a height-one arithmetic tower realising this Σ_1^A classification. We then set $\Gamma_{n_2,n_1}(A) = \Gamma_{n_1}^{1/n_2}(A)$. Exercise 1.11 implies that we have convergence to $\mathrm{Sp}(A)$ in the double limit $\lim_{n_2\to\infty}\lim_{n_1\to\infty}$. The Π_2^A classification follows from the fact that $\mathrm{Sp}(A) \subset \mathrm{Sp}_{1/n_2}(A)$ for each $n_2 \in \mathbb{N}$.

Exercise 3.9

For the first part, let $\{d_n\}$ be a null sequence with $\lim_{n\to\infty} c_n'/d_n = 0$. We simply run `CompSpec` and `PseudoSpec` with $\{c_n\}$ in the algorithms replaced by $\{d_n\}$. For large n we must have $D_{f,n}(A) \le d_n$ and, hence, obtain Δ_2^A classifications. Without $\{c_n\}$ or $\{c_n'\}$, `PseudoSpec` need not converge for the reasons outlined in the discussion just before Theorem 3.2.3.

Exercise 3.10

Take an arbitrary e_j, then $\|(I - \mathcal{P}_{f(n)}^*\mathcal{P}_{f(n)})Ae_j\| \le c_n$ for $n \ge j$. Hence,

$$\left\| Ae_j - \sum_{i=1}^{n} \langle Ae_j, e_i\rangle e_i \right\| \le c_n$$

and we can obtain arbitrarily accurate approximations of Be_j for $B = A$. A similar argument works for $B = A^*$. To deal with $B = A^2$, we can apply the same argument again with e_j replaced by $\sum_{i=1}^n \langle Ae_j, e_i\rangle e_i$, where we use that fact that

$$\left\| A^2e_j - A\left(\sum_{i=1}^{n} \langle Ae_j, e_i\rangle e_i\right) \right\| \le Mc_n.$$

It is now clear how to extend to general $B = \prod_{j=1}^n A_j$, where each A_j is either A or A^*. Once we have arbitrarily accurate approximations of Be_j for each e_j, we can clearly approximate the matrix entries.

Exercise 3.11

Suppose that $A \in \Omega_f$, $z \in \mathbb{C}$, $n \in \mathbb{N}$ and $\delta > 0$. Let $\epsilon = [\sigma_{\inf}(\mathcal{P}_{f(n)}(A - zI)\mathcal{P}_n^*) + \delta]^2$ and consider the matrix

$$B = [\mathcal{P}_{f(n)}(A - zI)\mathcal{P}_n^*]^*[\mathcal{P}_{f(n)}(A - zI)\mathcal{P}_n^*] - \epsilon I_n \in \mathbb{C}^{n\times n},$$

where I_n is the $n \times n$ identity matrix. B is a self-adjoint matrix and is not positive semi-definite. It follows that B can be put into the form $PBP^T = LDL^*$, where L is lower triangular with 1's along its diagonal, D is block diagonal with block sizes 1×1 or 2×2 and P is a permutation matrix. This can be computed in finitely many arithmetic operations and comparisons. Let x be an eigenvector of B with negative eigenvalue (B is not positive semi-definite) and set $y = L^*Px$. Note that

$$\langle y, Dy\rangle = \langle L^*Px, DL^*Px\rangle = \langle x, Bx\rangle < 0.$$

It follows that there exists a unit vector y_1 with $\langle y_1, Dy_1\rangle < 0$. Such a vector is easy to spot if a value in one of the 1×1 blocks of D is negative. If not, we need to consider 2×2 blocks. We can find a 2×2 block with a negative eigenvalue by computing the trace and determinant. Without loss of generality we assume that this block is the upper 2×2 portion of D. We can then find $y_1 = (a, b, 0, \dots, 0)^T \neq 0$ with $\langle y_1, Dy_1\rangle < 0$.

Since L^* is invertible and upper triangular, we can efficiently solve for $\tilde{x}_1 = P^T(L^*)^{-1}y_1$ using finitely many arithmetic operations and comparisons. We then approximately normalise $\tilde{x}_1$ by computing $\|\tilde{x}_1\| \approx t_1(\rho) > 0$ to precision $\rho > 0$ using finitely many arithmetic operations and comparisons. If we set $\hat{x}_1 = \tilde{x}_1/t_1(\rho)$ then

$$1 - \frac{\rho}{t_1(\rho)} = \frac{t_1(\rho) - \rho}{t_1(\rho)} \leq \|\hat{x}_1\| \leq \frac{t_1(\rho) + \rho}{t_1(\rho)} = 1 + \frac{\rho}{t_1(\rho)}.$$

So we successively choose ρ smaller until we reach ρ_1 such that $\rho_1/t_1(\rho_1) < \delta$. This is always possible since $\lim_{\rho\downarrow 0} t_1(\rho) = \|\tilde{x}_1\| > 0$. Let $t_1 = t_1(\rho_1)$, then

$$\langle \hat{x}_1, B\hat{x}_1\rangle = t_1^{-2}\langle L^*P\tilde{x}_1, DL^*P\tilde{x}_1\rangle = t_1^{-2}\langle y_1, Dy_1\rangle < 0.$$

Note that

$$\left\|\mathcal{P}_{f(n)}(A - zI)\mathcal{P}_n^*\hat{x}_1\right\|^2 = \langle \hat{x}_1, B\hat{x}_1\rangle + \|\hat{x}_1\|^2 \epsilon < \|\hat{x}_1\|^2\epsilon = \|\hat{x}_1\|^2[\sigma_{\inf}(\mathcal{P}_{f(n)}(A - zI)\mathcal{P}_n^*) + \delta]^2.$$

In exactly the same way by considering the adjoint, we can compute $\hat{x}_2$ such that

$$\left\|\mathcal{P}_{f(n)}(A - zI)^*\mathcal{P}_n^*\hat{x}_2\right\|^2 < \|\hat{x}_2\|^2[\sigma_{\inf}(\mathcal{P}_{f(n)}(A - zI)^*\mathcal{P}_n^*) + \delta]^2.$$

Let $J_1 = \|\mathcal{P}_{f(n)}(A - zI)\mathcal{P}_n^*\hat{x}_1\|^2/\|\hat{x}_1\|^2$ and $J_2 = \|\mathcal{P}_{f(n)}(A - zI)^*\mathcal{P}_n^*\hat{x}_2\|^2/\|\hat{x}_2\|^2$. If $J_1 > J_2$, we set $x_n = \mathcal{P}_n^*\hat{x}_2$, otherwise we set $x_n = \mathcal{P}_n^*\hat{x}_1$. It follows that

$$\min\{\|(A - zI)x_n\|, \|(A - zI)^*x_n\|\} \leq \min\left\{\|\mathcal{P}_{f(n)}(A - zI)x_n\|, \|\mathcal{P}_{f(n)}(A - zI)^*x_n\|\right\} + c_n\|x_n\| \leq [\Phi_n(z, A) + \delta]\|x_n\|.$$

Now suppose that z_n is in the output of `CompSpec` and converges to z as $n \to \infty$, where z is an isolated part of the spectrum whose Riesz projection has finite rank. Suppose also that

$$\|(A - zI)x_n\| \leq [\Phi_n(z, A) + \delta]\|x_n\|$$

(we can argue using dual spaces if not). Let $v_n = x_n/\|x_n\|$. We also replace δ with a sequence $\delta_n \downarrow 0$ to ensure that $\|(A - zI)v_n\| \to 0$. Let E be the eigenspace associated with z. We may write $v_n = e_n + w_n$ where $e_n \in E$ and $w_n \in E^\perp$. We have $(A - zI)$ is bounded below on $E^\perp$ and, hence, $\lim_{n\to\infty} w_n = 0$. It follows that $\mathrm{dist}(v_n, E) \to 0$.

Exercise 3.12

It is enough to prove the upper bound for $\Omega_{l^2(\mathbb{N})}$ and the lower bounds for Ω_g. We begin with the upper bound. Arguing as in the solution to Exercise 1.15, for $z \in \mathbb{C}$ and $A \in \Omega_{l^2(\mathbb{N})}$, let $\tilde{\gamma}_{n_2,n_1}(z, A)$ be an approximation of

$$\gamma_{n_2,n_1}(z, A) = \min\left\{\sigma_{\inf}(\mathcal{P}_{n_1}(A - zI)\mathcal{P}_{n_2}^*), \sigma_{\inf}(\mathcal{P}_{n_1}(A^* - \bar{z}I)\mathcal{P}_{n_2}^*)\right\}$$

that is computed in finitely many arithmetic operations and comparisons so that

$$\gamma_{n_2,n_1}(z, A) - \frac{1}{n_1} \leq \tilde{\gamma}_{n_2,n_1}(z, A) \leq \gamma_{n_2,n_1}(z, A).$$

For each $A \in \Omega_{l^2(\mathbb{N})}$ and $K \in \mathscr{K}(\mathbb{C})$, we set

$$\Gamma_{n_3,n_2,n_1}(A,K) = \text{“Is } \min_{z \in K_{n_3}} \tilde{\gamma}_{n_2,n_1}(z,A) + \frac{1}{n_2} \leq 2^{1-n_3}\text{?”}$$

It is easy to see that

$$\lim_{n_2\to\infty}\lim_{n_1\to\infty} \Gamma_{n_3,n_2,n_1}(A,K) = \text{“Is } \min_{z \in K_{n_3}} \gamma(z,A) < 2^{1-n_3}\text{?”}$$

The Π_3^A classification now follows from the same arguments in the proof of Theorem 3.3.14.

For the lower bound for Ω_g, it is enough to consider the following problem for self-adjoint operators $A \in \Omega_{\mathrm{SA}}$:

$$\Xi : A \to \text{“Is } 0 \in \mathrm{Sp}(A)\text{?”}$$

Our proof strategy will be to embed a combinatorial problem from Chapter 2. Let Ω' denote the collection of all infinite matrices $a = \{a_{i,j}\}_{i,j\in\mathbb{N}}$ with entries $a_{i,j} \in \{0,1\}$. Recall the problem function

$$\Xi_{2,Q}(\{a_{i,j}\}) = \begin{cases} 1, & \text{if } \sum_i a_{i,j} = \infty \text{ for all but finitely many } j, \\ 0, & \text{otherwise.} \end{cases}$$

In Chapter 2, we showed that $\{\Xi_{2,Q}, \Omega', \mathcal{M}_{\mathrm{dec}}, \Lambda'\} \notin \Delta_3^G$, where Λ' is the set of component-wise evaluations of $\{a_{i,j}\}$.

Suppose for a contradiction that $\{\Gamma_{n_2,n_1}\}$ is a Δ_3^G-tower of algorithms for $\{\Xi, \Omega_{\mathrm{SA}}, \mathcal{M}_{\mathrm{dec}}, \Lambda\}$. Given a sequence $\{c_i\}_{i\in\mathbb{N}}$ with each $c_i \in \{0,1\}$, list the indices where $c_i = 1$ as

$$\{i : c_i = 1\} = \{i_1, i_2, \ldots\}, \quad i_1 \leq i_2 \leq \ldots.$$

This set may be empty, finite, or countably infinite. Define $i_0 = 0$ and let $r(\{c_i\})$ be one greater than the cardinality of $\{i : c_i = 1\}$. If $r(\{c_i\}) < \infty$, then we define

$$l_k = 2 + i_k - i_{k-1}, \quad k = 1, \ldots, r(\{c_i\}) - 1, \quad l_{r(\{c_i\})} = \infty,$$

otherwise we define

$$l_k = 2 + i_k - i_{k-1}, \quad k \in \mathbb{N}.$$

We set

$$C(\{c_i\}) = \bigoplus_{k=1}^{r(\{c_i\})} A_{l_k}, \quad A_m = \begin{pmatrix} 1 & & & & 1 \\ & 0 & & & \\ & & \ddots & & \\ & & & 0 & \\ 1 & & & & 1 \end{pmatrix} \in \mathbb{C}^{m\times m},$$

where $A_\infty = \mathrm{diag}(1,0,0,\ldots)$. Note that $\mathrm{Sp}(C(\{c_i\})) \subset \{0,1,2\}$ and $1 \in \mathrm{Sp}(C(\{c_i\}))$ if and only if $r(\{c_i\}) < \infty$. Given $\{a_{i,j}\} \in \Omega'$, define the self-adjoint operator

$$A = A(\{a_{i,j}\}) = \bigoplus_{j=1}^{\infty} \left[C(\{a_{i,j}\}_{i\in\mathbb{N}}) - (1 - 2^{-(1+j)})I\right] \in \Omega_{\mathrm{SA}}.$$

If $\Xi_{2,Q}(\{a_{i,j}\}) = 1$, then $0 \notin \mathrm{Sp}(A)$. If $\Xi_{2,Q}(\{a_{i,j}\}) = 0$, there exists a null sequence in $\mathrm{Sp}(A)$ and $0 \in \mathrm{Sp}(A)$. We set

$$\widehat{\Gamma}_{n_2,n_1}(\{a_{i,j}\}) = 1 - \Gamma_{n_2,n_1}(A(\{a_{i,j}\})).$$

This defines a general algorithm that uses Λ'. Furthermore, the convergence of Γ_{n_2,n_1} implies that

$$\lim_{n_2\to\infty}\lim_{n_1\to\infty} \widehat{\Gamma}_{n_2,n_1}(\{a_{i,j}\}) = \Xi_{2,Q}(\{a_{i,j}\}).$$

However, this contradicts $\{\Xi_{2,Q}, \Omega', \mathcal{M}_{\mathrm{dec}}, \Lambda'\} \notin \Delta_3^G$.

Exercise 3.13

We first prove that for every $\epsilon > 0$, $\{S_\epsilon, \Omega_{\mathrm{D}}, \mathcal{M}_{\mathrm{H}}, \Lambda\} \in \Delta_2^A$. Given $A = \mathrm{diag}(a_1, a_2, \ldots) \in \Omega_{\mathrm{D}}$, we let $\Gamma_n(A)$ be an approximation of

$$\{z \in \mathbb{C} : \mathrm{dist}(z, \{a_1, \ldots, a_n\}) = \epsilon\}$$

to an accuracy of $1/n$ in the Hausdorff metric. If $z \in S_\epsilon(A)$, then there exists $w \in \mathrm{Sp}(A)$ with $|z - w| = \epsilon$ and there are no points in $\mathrm{Sp}(A)$ closer to z (though there may be ties). It follows that $\mathrm{dist}(z, \{a_1, \ldots, a_n\})$ converges monotonically down to ϵ. A simple geometric argument then shows that $\lim_{n\to\infty} \mathrm{dist}(z, \Gamma_n(A)) = 0$. On the other hand, suppose that $z_{n_j} \in \Gamma_{n_j}(A)$ for a subsequence $\{n_j\}$ and that $z_{n_j} \to z$. Then clearly $\mathrm{dist}(z, \mathrm{Sp}(A)) = \epsilon$. The Δ_2^A upper bound now follows. The proof that $\{\mathrm{Sp}, \Omega_{\mathrm{D}}, \mathcal{M}_{\mathrm{H}}, \Lambda\} \notin \Pi_1^G$ carries over straightforwardly to show that $\{S_\epsilon, \Omega_{\mathrm{D}}, \mathcal{M}_{\mathrm{H}}, \Lambda\} \notin \Pi_1^G$. To prove that $\{S_\epsilon, \Omega_{\mathrm{D}}, \mathcal{M}_{\mathrm{H}}, \Lambda\} \notin \Sigma_1^G$, we note that we can alter the argument by adding points to the spectrum that remove points of $S_\epsilon(A)$ close to the real line.

Let $z \in \mathbb{C}$ be fixed. Given $A \in \Omega_{\mathrm{D}}$, let

$$\tilde{\Gamma}_{n_2,n_1}(A) = \text{"Is } \mathrm{dist}(z, \{a_1, \ldots, a_{n_1}\}) \in [\epsilon(1 - 2^{-n_2}), \epsilon(1 + 2^{-n_2}))\text{?"}$$

Then, owing to the fact the interval in the decision problem is closed at its left endpoint and open at its right endpoint and that $\mathrm{dist}(z, \{a_1, \ldots, a_{n_1}\})$ converges down to $\mathrm{dist}(z, \mathrm{Sp}(A))$,

$$\lim_{n_1\to\infty} \tilde{\Gamma}_{n_2,n_1}(A) = \text{"Is } \mathrm{dist}(z, \mathrm{Sp}(A)) \in [\epsilon(1 - 2^{-n_2}), \epsilon(1 + 2^{-n_2}))\text{?"}$$

It is now clear that $\{\tilde{\Gamma}_{n_2,n_1}\}$ provides the desired Π_2^A algorithm for $\{\Xi_3, \Omega_{\mathrm{D}}, \mathcal{M}_{\mathrm{dec}}, \Lambda\}$. The proof that $\{\Xi_3, \Omega_{\mathrm{D}}, \mathcal{M}_{\mathrm{dec}}, \Lambda\} \notin \Delta_2^G$ follows from Step 1 of the proof of Theorem 3.3.14 by setting $z = -\epsilon$. Other values of z can be dealt with by shifting the argument with multiples of the identity operator.

Exercise 3.14

Using the alternative form of the total variation, we have

$$\mathrm{TV}_{[-r,r]^d}(\psi_{\mathbf{m}}) = \sum_{k=1}^{d} \sum_{1\le i_1<\ldots<i_k\le d} \int_{-r}^{r} \cdots \int_{-r}^{r} \left| \frac{\partial^k \psi_{\mathbf{m}}}{\partial x_{i_1} \cdots \partial x_{i_k}}(\tilde{x}) \right| \mathrm{d}x_{i_1} \cdots \mathrm{d}x_{i_k},$$

where $\tilde{x}$ has $\tilde{x}_j = x_j$ for $j = i_1, ..., i_k$ and $\tilde{x}_j = r$ otherwise. We can use the recurrence relations for Hermite functions as well as Cramér's inequality ($|\psi_n(x)| \le \pi^{-1/4} \le 1$) to gain the bound

$$\int_{-r}^{r} \cdots \int_{-r}^{r} \left| \frac{\partial^k \psi_{\mathbf{m}}}{\partial x_{i_1} \cdots \partial x_{i_k}}(\tilde{x}) \right| \mathrm{d}x_{i_1} \cdots \mathrm{d}x_{i_k} \le \left(2r\sqrt{2(\|\mathbf{m}\|_{\ell^\infty} + 1)}\right)^k.$$

It follows that

$$\mathrm{TV}_{[-r,r]^d}(\psi_{\mathbf{m}}) \le \sum_{k=1}^{d} \left(2r\sqrt{2(\|\mathbf{m}\|_{\ell^\infty} + 1)}\right)^k \sum_{1\le i_1<\cdots<i_k\le d} 1 = \sum_{k=1}^{d} \left(2r\sqrt{2(\|\mathbf{m}\|_{\ell^\infty} + 1)}\right)^k \binom{d}{k} = \left(1 + 2r\sqrt{2(\|\mathbf{m}\|_{\ell^\infty} + 1)}\right)^d - 1.$$

Exercise 3.15

By taking a complex conjugation of the eigenvalue equation, we see that $u_j^* = \overline{u_j}$ (up to normalisation). The form of $\|Q_j\|$ now follows. Code for the imaginary cubic oscillator can be found in "chapter3/imaginary_cubic.m" in the repository. This exponential blow-up of the condition number means we need extended precision for large eigenvalues.

Exercise 3.16

The space $L^2(\mathbb{R})$ decomposes into a (non-orthogonal) direct sum of closed, H-invariant subspaces

$$L^2(\mathbb{R}) = \mathrm{ran}(Q_1) \oplus \cdots \oplus \mathrm{ran}(Q_m) \oplus \mathrm{ran}(I - \mathcal{P}_m),$$

where $\mathcal{P}_m = \sum_{j=1}^m Q_j$. Using the fact that each Q_j commutes with H, we have

$$(H - zI)^{-1} = (I - \mathcal{P}_m)(H - zI)^{-1}(I - \mathcal{P}_m) + \sum_{j=1}^{m}(\lambda_j - z)^{-1}Q_j.$$

It follows that

$$\|(H - zI)^{-1}\| \le \left(1 + \sum_{j=1}^{m} \|Q_j\|\right) \|(H - zI)\restriction^{-1}_{\mathrm{ran}(I-\mathcal{P}_m)}\| + \sum_{j=1}^{m} \frac{\|Q_j\|}{|z - \lambda_j|}. \tag{9}$$

Using the lemma in the exercise, we observe that

$$\|e^{-tH}\restriction_{\mathrm{ran}(I-\mathcal{P}_m)}\| = \lim_{j\to\infty} \|e^{-tH}\restriction_{\mathrm{ran}(Q_{m+1})\oplus\cdots\oplus\mathrm{ran}(Q_j)}\| \le e^{-t\lambda_{m+1}} \sum_{j=m+1}^{\infty} e^{-t(\lambda_j-\lambda_{m+1})}\|Q_j\|.$$

We assume that $\|Q_j\| \le \exp(\frac{j\pi}{\sqrt{3}})$ and use the fact that $\lambda_j - \lambda_{m+1} \ge (j - (m + 1))(\pi/\sqrt{3} + 1)$ for $j \ge m + 1$. For $t \ge 1$,

$$\begin{aligned}
\sum_{j=m+1}^{\infty} e^{-t(\lambda_j-\lambda_{m+1})}\|Q_j\| &\le \sum_{j=m+1}^{\infty} e^{-(\lambda_j-\lambda_{m+1})}\|Q_j\| \\
&\le \sum_{j=m+1}^{\infty} e^{-(j-(m+1))(\pi/\sqrt{3}+1)} \exp(j\pi/\sqrt{3}) \\
&= \exp\left((m + 1)\pi/\sqrt{3}\right) \sum_{j=m+1}^{\infty} e^{m+1-j} = \frac{e}{e-1} \exp\left((m + 1)\pi/\sqrt{3}\right).
\end{aligned}$$

We must also consider the case that $t < 1$. Here, we use the facts that $\|e^{-tH}\restriction_{\mathrm{ran}(I-\mathcal{P}_m)}\| \le 1$ and $\lambda_{m+1} \le c(m + 1)^{6/5}$ with $c = [2\Gamma(\frac{11}{6})\sqrt{\pi}/(\sqrt{3}\Gamma(\frac{4}{3}))]^{6/5}$ to see that if $t < 1$, then

$$\|e^{-tH}\restriction_{\mathrm{ran}(I-\mathcal{P}_m)}\| \le 1 \le e^{-\lambda_{m+1}}e^{c(m+1)^{6/5}} \le e^{-t\lambda_{m+1}}e^{c(m+1)^{6/5}}.$$

It follows that, for $t \ge 0$,

$$\|e^{-tH}\restriction_{\mathrm{ran}(I-\mathcal{P}_m)}\| \le e^{-t\lambda_{m+1}} \max\left\{\frac{e}{e-1} \exp\left((m + 1)\pi/\sqrt{3}\right), e^{c(m+1)^{6/5}}\right\} \le e^{-t\lambda_{m+1}}e^{c(m+1)^{6/5}}.$$

The Hille–Yosida theorem now implies that

$$\|(H - zI)\restriction^{-1}_{\mathrm{ran}(I-\mathcal{P}_m)}\| \le \frac{e^{c(m+1)^{6/5}}}{\lambda_{m+1} - \mathrm{Re}(z)} \quad \forall z \in \mathbb{C} \text{ with } \mathrm{Re}(z) < \lambda_{m+1}.$$

Using $\|Q_j\| \le \exp(\frac{j\pi}{\sqrt{3}})$ again, we see that if $\mathrm{Re}(z) \le \lambda_m$, then

$$\left(1 + \sum_{j=1}^{m} \|Q_j\|\right) \|(H - zI)\restriction^{-1}_{\mathrm{ran}(I-\mathcal{P}_m)}\| \le \frac{e^{(m+1)\pi/\sqrt{3}} - 1}{e^{\pi/\sqrt{3}} - 1} \cdot \frac{e^{c(m+1)^{6/5}}}{\pi/\sqrt{3} + 1}.$$

Combining with (9), we see that if $\lambda_{m-1} \le \mathrm{Re}(z) \le \lambda_m$ with $z \notin \mathrm{Sp}(H)$, then

$$\begin{aligned}
\|(H - zI)^{-1}\| &\le \frac{\|Q_{m-1}\|}{|\lambda_{m-1} - z|} + \frac{\|Q_m\|}{|\lambda_m - z|} + \frac{1}{\pi/\sqrt{3} + 1}\left[\frac{e^{(m-1)\pi/\sqrt{3}} - 1}{e^{\pi/\sqrt{3}} - 1} - 1 + \frac{e^{(m+1)\pi/\sqrt{3}} - 1}{e^{\pi/\sqrt{3}} - 1} \cdot e^{c(m+1)^{6/5}}\right] \\
&\le \frac{\|Q_{m-1}\|}{|\lambda_{m-1} - z|} + \frac{\|Q_m\|}{|\lambda_m - z|} + \frac{\exp[(m + 1)\pi/\sqrt{3} + c(m + 1)^{6/5}]}{14}.
\end{aligned}$$

Together with $\|Q_j\| \le \exp(\frac{j\pi}{\sqrt{3}})$, this provides the desired result.

4 Chapter 4

Exercise 4.1

For the first part, μ_v is clearly positive and by definition

$$\mu_v(\mathbb{C}) = \langle \mathcal{E}(\mathbb{C})v, v\rangle = \langle v, v\rangle = \|v\|^2.$$

If $z \notin \mathrm{Sp}(A)$, then there exists an open set U containing z with $U \cap \mathrm{Sp}(A) = \emptyset$. By the spectral theorem for normal operators, $\mathcal{E}(U) = 0$ and, hence, $\mu_v(U) = 0$ and $z \notin \mathrm{supp}(\mu_v)$. Note that $\mathrm{supp}(\mu_v)$ need not be the whole of $\mathrm{Sp}(A)$.

Now suppose for a contradiction that there exists

$$z \in \mathrm{Sp}(A) \setminus \mathrm{Cl}\left(\bigcup_{v\in\mathcal{H}} \mathrm{supp}(\mu_v)\right).$$

There exists an open set U that contains z and does not intersect $\mathrm{supp}(\mu_v)$ for all $v \in \mathcal{H}$. Hence, $\mu_v(U) = 0$ for all v, and hence $\mathcal{E}(U) = 0$. However, this contradicts the spectral theorem.

For the final part, note that if P, Q are projections with $PQ = 0$, we have $\mathrm{ran}(Q) \subset \ker(P) = (\mathrm{ran}(P))^\perp$. Hence, $Qu \perp Pu$ for each $u \in \mathcal{H}$. In particular, if $U \cap V = \emptyset$, we have $\mathcal{E}(U)\mathcal{E}(V) = \mathcal{E}(U \cap V) = 0$, so $\mathcal{E}(U)u \perp \mathcal{E}(V)u$ for each $u \in \mathcal{H}$. Let $\{S_j\}$ be a measurable partition of $\mathbb{C}$, $v, w \in \mathcal{H}$ and $\alpha_j \in \mathbb{C}$ with $|\alpha_j| = 1$ such that $|\langle \mathcal{E}(S_j)v, w\rangle| = \alpha_j\langle \mathcal{E}(S_j)v, w\rangle = \langle \alpha_j\mathcal{E}(S_j)v, w\rangle$. We then have

$$\sum_{j=1}^{\infty} |\mu_{v,w}(S_j)| = \left\langle \sum_{j=1}^{\infty} \alpha_j\mathcal{E}(S_j)v, w\right\rangle \leq \left\|\sum_{j=1}^{\infty} \alpha_j\mathcal{E}(S_j)v\right\| \|w\|$$

Since $S_i \cap S_j = \emptyset$ for $i \neq j$, we have $\alpha_i\mathcal{E}(S_i)u \perp \alpha_j\mathcal{E}(S_j)u$ for each $u \in \mathcal{H}$, and, hence,

$$\left\|\sum_{j=1}^{\infty} \alpha_j\mathcal{E}(S_j)v\right\|^2 = \sum_{j=1}^{\infty} \|\mathcal{E}(S_j)v\|^2 = \sum_{j=1}^{\infty} \mu_v(S_j) = \mu_v(\mathbb{C}) = \|v\|^2.$$

Hence,

$$\sum_{j=1}^{\infty} |\mu_{v,w}(S_j)| \leq \|v\|\|w\|.$$

The result follows since $\{S_j\}$ was arbitrary.

Exercise 4.2

Squaring the given inequality and using the spectral theorem, we have:

$$\int_{\mathbb{R}} |\lambda - \lambda_0|^2 \,\mathrm{d}\mu_v(\lambda) \leq \epsilon^2\|v\|^2.$$

Applying Markov's inequality we then have:

$$\mu_v(\{\lambda \in \mathbb{R} : |\lambda - \lambda_0| \geq \delta\}) = \mu_v(\{\lambda \in \mathbb{R} : |\lambda - \lambda_0|^2 \geq \delta^2\}) \leq \frac{1}{\delta^2}\int_{\mathbb{R}} |\lambda - \lambda_0|^2 \,\mathrm{d}\mu_v(\lambda).$$

Hence we get:

$$\mu_v(\{\lambda \in \mathbb{R} : |\lambda - \lambda_0| \geq \delta\}) \leq \frac{\epsilon^2}{\delta^2}\|v\|^2.$$

Since $\{\lambda \in \mathbb{R} : |\lambda - \lambda_0| \geq \delta\}^c = (\lambda_0 - \delta, \lambda_0 + \delta)$ and $\mu_v(\mathbb{R}) = \|v\|^2$, we then obtain:

$$\mu_v((\lambda_0 - \delta, \lambda_0 + \delta)) \geq \|v\|^2 - \frac{\epsilon^2}{\delta^2}\|v\|^2 = \left[1 - \frac{\epsilon^2}{\delta^2}\right]\|v\|^2.$$

Then if $\delta = \sqrt{\epsilon}$ say, we will have $\mu_v((\lambda_0-\delta, \lambda_0+\delta)) \geq (1-\epsilon)\|v\|^2$. For $1-\epsilon \approx 1$ we will then have $\mu_v((\lambda_0-\delta, \lambda_0+\delta)) \approx \|v\|^2$, meaning that most of the measure is concentrated on $(\lambda_0 - \delta, \lambda_0 + \delta)$.

Exercise 4.3

Suppose that A is a bounded normal operator and that v is a generating vector for A. Let $R > 0$ be such that $\mathrm{Sp}(A) \subset B_{R/2}(0)$. Fix a Borel subset $S \subset B_R(0)$ and let p be a polynomial in z and $\bar{z}$. Then

$$\|\chi_S(A)v - p(A, A^*)v\|^2 = \int_{\mathrm{Sp}(A)} |\chi_S(z) - p(z, \bar{z})|^2 \,\mathrm{d}\mu_v(z).$$

Since μ is finite and compactly supported, the space of polynomials in z and $\bar{z}$ is dense in $L^2(\mu)$. It follows that $\chi_S(A)v$ lies in $\mathrm{Cl}(\mathrm{span}\{A^{*k}A^l v : k, l \in \mathbb{Z}_{\geq 0}\})$ and hence v is a star-cyclic vector for A.

For the example showing that this need not hold for unbounded operators, let A be the multiplication operator M_μ, where μ is the measure on $\mathbb{R}_{>0}$ with density given in the question. Let $v = 1$, which is a generating vector for A. It is easily checked that A is self-adjoint. Using the change of variables $y = \log(x)$, we have for $n \in \mathbb{Z}_{\geq 0}$,

$$\int_{\mathbb{R}_{>0}} x^n \sin(2\pi \log(x)) \,\mathrm{d}\mu(x) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2e\pi}} \exp(-y^2/2 + (n+1)y) \sin(2\pi y) \,\mathrm{d}y.$$

A further affine change of variables and the periodicity of sin implies that this integral vanishes. It follows that

$$\langle A^n v, \sin(2\pi \log(A))v \rangle = \int_{\mathbb{R}_{>0}} x^n \sin(2\pi \log(x)) \,\mathrm{d}\mu(x) = 0.$$

Hence, $\sin(2\pi \log(A))v$ is a nonzero vector that is orthogonal to the space $\mathrm{Cl}(\mathrm{span}\{A^{*k}A^l v : k, l \in \mathbb{Z}_{\geq 0}\})$, so v cannot be a star-cyclic vector.

Now suppose that v is a generating vector for a (possibly unbounded) normal operator A. Let $f \in C_c^\infty(\mathbb{R}^2)$. We will show that its Hermite function expansion converges uniformly. Let

$$P_N f = \sum_{|\mathbf{m}| \leq N} \langle f, \psi_{\mathbf{m}} \rangle \psi_{\mathbf{m}}.$$

Since $P_N f$ converges in $L^2(\mathbb{R}^2)$, it has a subsequence $P_{N_k} f$ that converges almost everywhere. However, we know from the proof of Proposition 3.4.7 (in fact, the easy extension to two dimensions) that the coefficients $\langle f, \psi_{\mathbf{m}} \rangle$ decay faster than every inverse power of $|\mathbf{m}|$. Since the Hermite functions are uniformly bounded, it follows that $\{P_N f(x)\}$ is uniformly Cauchy in x. The uniform convergence of $P_N f$ to f follows. Writing $z = x + \mathrm{i}y$, we have

$$\mathcal{H}_G := \mathrm{Cl}(\mathrm{span}\{e^{-|z|^2/2} p(x, y) : \text{polynomials } p\}) = \mathrm{Cl}(\mathrm{span}\{e^{-|z|^2/2} p(z, \bar{z}) : \text{polynomials } p\}).$$

Since each compactly supported function can be uniformly approximated by a smooth one, it follows that $C_c(\mathbb{C}) \subset \mathcal{H}_G$. The result in the hint now implies that $\mathcal{H}_G = L^2(\mu)$. Hence, we may extend the argument above to conclude that $\exp(-|A|^2/2)v$ is a star-cyclic vector for A.

The first part of the multiplication operator version of the spectral theorem is standard, and the construction of the isomorphism is clear. For the second part, we first pick a vector $v \in \mathcal{D}(A^{*k}A^l)$ for all $k, l \in \mathbb{Z}_{\geq 0}$. Using the above, we can produce a star-cyclic vector for $\mathrm{Cl}(\mathrm{span}\{\chi(S)v : \text{Borel subset } S \subset \mathbb{C}\})$, which is a reducing subspace for $\mathcal{H}$. We continue this process inductively to obtain the direct sum.

Exercise 4.4

Suppose that $(A_n - z_0 I)^{-1} \xrightarrow{gs} (A - z_0 I)^{-1}$ for some $z_0 \in \mathbb{C} \setminus \mathbb{R}$. Following the hint, if $|z - z_0| < |\mathrm{Im}(z_0)|$ then, by summing a geometric series,

$$\sum_{k=0}^{\infty} (z - z_0)^k (A - z_0 I)^{-(k+1)} \mathcal{P} = (A - zI)^{-1} \mathcal{P}.$$

We can apply a similar argument to A_n to arrive at

$$(A - zI)^{-1}\mathcal{P} - (A_n - zI)^{-1}\mathcal{P}_n = \sum_{k=0}^{\infty} (z - z_0)^k \left[(A - z_0 I)^{-(k+1)} \mathcal{P} - (A_n - z_0 I)^{-(k+1)} \mathcal{P}_n \right]. \tag{10}$$

The tail of the series is uniformly bounded as we increase n since

$$\left\|(A-z_0I)^{-(k+1)}\mathcal{P}-(A_n-z_0I)^{-(k+1)}\mathcal{P}_n\right\| \leq \left\|(A-z_0I)^{-(k+1)}\right\|+\left\|(A_n-z_0I)^{-(k+1)}\right\| \leq 2/|\mathrm{Im}(z_0)|^{k+1}.$$

Since each of the terms in the sum on the right-hand side of (10) converges strongly to zero, we must have $(A_n-zI)^{-1} \xrightarrow{gs} (A-zI)^{-1}$ as $n\to\infty$. This argument gives convergence for all z whose imaginary part has the same sign as that of z_0. It remains to show that convergence holds for $\overline{z}_0$. To that end, we note that $(A_n-z_0I)^{-1} \xrightarrow{gw} (A-z_0I)^{-1}$. Upon taking adjoints and inner products, we see that $(A_n-\overline{z_0}I)^{-1} \xrightarrow{gw} (A-\overline{z_0}I)^{-1}$. Moreover, from normality, we have that

$$\|(A_n-\overline{z_0}I)^{-1}\mathcal{P}_nv\| = \|(A_n-z_0I)^{-1}\mathcal{P}_nv\| \to \|(A-z_0I)^{-1}\mathcal{P}v\| = \|(A-\overline{z_0}I)^{-1}\mathcal{P}v\| \quad \forall v\in\mathcal{K}.$$

Hence, $(A_n-\overline{z_0}I)^{-1} \xrightarrow{gs} (A-\overline{z_0}I)^{-1}$.

Exercise 4.5

We first note that $\|A_N\|\leq\|A\|\leq 1$, and, hence, $\mathrm{Sp}(A_N)\subset[-1,1]$. We have

$$\int_{\mathbb{R}} y^k \,\mathrm{d}\mu_{N;v}(y) = \langle A_N^k v, v\rangle.$$

Since $v, Av\in\mathrm{ran}(\mathcal{P}_{V_N})$, we have $A_Nv=Av$. Iterating, we have $A_N^kv=A^kv$ for $k\in\mathbb{Z}_{\geq 0}$. So

$$\langle A_N^kv,v\rangle = \langle A^kv,v\rangle = \int_{\mathbb{R}} y^k\,\mathrm{d}\mu_v(y).$$

Let $\phi:[-1,1]\to\mathbb{R}$ be Lipschitz with $\|\phi\|_{C^{0,1}([-1,1])}\leq 1$. Let $\epsilon>0$ and take $p\in P_N$ such that

$$\|\phi-p\|_{L^\infty([-1,1])} \leq \frac{\pi}{2(N+1)}+\epsilon.$$

We have $\int_{\mathbb{R}} p(y)\,\mathrm{d}\mu_{N;v}(y) = \int_{\mathbb{R}} p(y)\,\mathrm{d}\mu_v(y)$ from the first part. Hence,

$$\int_{\mathbb{R}} \phi(y)\,\mathrm{d}(\mu_{N;v}-\mu_v)(y) = \int_{\mathbb{R}} (\phi(y)-p(y))\,\mathrm{d}(\mu_{N;v}-\mu_v)(y)$$

and so

$$\left|\int_{\mathbb{R}} \phi(y)\,\mathrm{d}(\mu_{N;v}-\mu_v)(y)\right| \leq \|\phi-p\|_{L^\infty([-1,1])}(\|\mu_{N;v}\|+\|\mu_v\|) \leq \frac{\pi\|v\|^2}{N+1}+2\epsilon\|v\|^2 = \frac{\pi}{N+1}+2\epsilon.$$

Taking the supremum over ϕ and then ϵ arbitrarily small, we have $d_{\mathrm{BL}^*}(\mu_v,\mu_{N;v})\leq\frac{\pi}{N+1}$.

Exercise 4.6

Since ϕ is smooth and compactly supported, it is α-Lipschitz for all $0<\alpha<1$. We define the measure $\mu(S)=\int_S\phi\,\mathrm{d}\lambda$, which has Radon–Nikodym derivative ϕ. Note that the proof of Theorem 4.3.15 does not really rely on μ_v being the spectral measure of an operator, it just relies on convergence of particular integrals, so we are granted that:

$$\left|\frac{1}{2\pi i}\left(\int_{\mathbb{R}}\frac{\mathrm{d}\mu(x)}{x-(x_0+\mathrm{i}\epsilon)}-\int_{\mathbb{R}}\frac{\mathrm{d}\mu(x)}{x-(x_0-\mathrm{i}\epsilon)}\right)-\phi(x_0)\right| \leq C\epsilon^{1/2}$$

for each $x_0\in\mathbb{R}$. The constant C in the proof of Theorem 4.3.15 depends only on the Lipschitz and supremum bounds of ϕ (which can be taken to be uniform) and the interval length η, so can be taken independent of x_0.

Now suppose that ϕ is supported in $[a,b]$. We have

$$\int_{\mathbb{R}}\frac{\mathrm{d}\mu(x)}{x-(x_0\pm\mathrm{i}\epsilon)} = \int_a^b\frac{\mathrm{d}\mu(x)}{x-(x_0\pm\mathrm{i}\epsilon)}.$$

It is enough to show that we can approximate each of these integrals uniformly in x_0 using Riemann sums. We deal with the case of '+' in the denominator. We note that $x\mapsto 1/(x-x_0+\mathrm{i}\epsilon)$ is Lipschitz and its derivative is bounded above by c/ϵ^2 for some constant c. In particular, the functions $f_{x_0}(x) = \phi(x)/(x - x_0 + i\epsilon)$ have a first derivative that is bounded independently of x_0. Hence, given $\delta > 0$, for sufficiently large n (independent of x_0) we have

$$\left|f_{x_0}(a + k(b-a)/n) - f_{x_0}(y)\right| < \delta$$

for all $y \in [a + (k-1)(b-a)/n, a + k(b-a)/n]$. Integrating over this interval we have

$$\left|\frac{(b-a)f_{x_0}(a + \frac{k}{n}(b-a))}{n} - \int_{a+(k-1)(b-a)/n}^{a+k(b-a)/n} f_{x_0}(y)\,\mathrm{d}y\right| < \frac{(b-a)\delta}{n}.$$

Summing over k, we obtain

$$\left|\frac{b-a}{n}\sum_{k=1}^{n} f_{x_0}\left(a + \frac{k}{n}(b-a)\right) - \int_a^b f_{x_0}(y)\,\mathrm{d}y\right| < (b-a)\delta.$$

This holds uniformly in x_0, so we have proved the first statement in the exercise. For the final part, we note that each continuous function that vanishes at infinity can be uniformly approximated by smooth compactly supported functions.

Exercise 4.7

Using Stone's formula, we have for all intervals $I \subset (a,b)$ that $\mu_v(I) \leq \frac{C}{\pi}|I|$. It follows that this bound holds with I replaced by any Borel set $S \subset (a,b)$. The results now easily follow.

Exercise 4.8

Let $p(x) = x^3 - x$ and let each function p_k^{-1}, $k = 1, 2$, be a branch of the inverse of $p(x)$ with domain $[-2\sqrt{3}/9,\ 2\sqrt{3}/9] = p([-1,1])$. For $n \in \mathbb{Z}_{\geq 0}$, a change of variables $\lambda = p(x)$ shows that

$$\langle A^n f, g\rangle = \int_{-1}^{1} [p(x)]^n f(x)\overline{g(x)}\,\mathrm{d}x = \int_{p([-1,1])} \lambda^n \sum_{k=1}^{2}\left[\frac{f(p_k^{-1}(\lambda))\overline{g(p_k^{-1}(\lambda))}}{|p'(p_k^{-1}(\lambda))|}\right]\mathrm{d}\lambda.$$

Since moments determine a measure on a bounded interval, this shows that $\mu_{f,g}$ is absolutely continuous with Radon–Nikodym derivative

$$\rho_{f,g}(\lambda) = \sum_{k=1}^{2}\left[\frac{f(p_k^{-1}(\lambda))\overline{g(p_k^{-1}(\lambda))}}{|p'(p_k^{-1}(\lambda))|}\right] \qquad \text{for almost every} \qquad \lambda \in \mathrm{Sp}(A) = p([-1,1]).$$

Exercise 4.9

Let $z \in \mathbb{C} \setminus \mathbb{R}$. Working in the Fourier domain, we see that

$$\langle (A - zI)^{-1} f, g\rangle = \int_{\mathbb{R}} (16\pi^4 k^4 - z)^{-1}\hat{f}(k)\overline{\hat{g}(k)}\,\mathrm{d}k = \int_0^\infty (16\pi^4 k^4 - z)^{-1}\hat{f}(-k)\overline{\hat{g}(-k)}\,\mathrm{d}k + \int_0^\infty (16\pi^4 k^4 - z)^{-1}\hat{f}(k)\overline{\hat{g}(k)}\,\mathrm{d}k.$$

We now perform a change of variables $\lambda = 16\pi^4 k^4$ to see that

$$\langle (A - zI)^{-1} f, g\rangle = \sum_{j=1}^{2}\int_0^\infty \frac{1}{8\pi\lambda^{3/4}}(\lambda - z)^{-1}\hat{f}\left((-1)^j\frac{\lambda^{1/4}}{2\pi}\right)\overline{\hat{g}\left((-1)^j\frac{\lambda^{1/4}}{2\pi}\right)}\mathrm{d}\lambda.$$

Since a finite measure on $\mathbb{R}$ is determined by its integration against rational functions with simple poles in the complex plane, $\mu_{f,g}$ is absolutely continuous with Radon–Nikodym derivative

$$\rho_{f,g}(\lambda) = \sum_{j=1}^{2}\frac{1}{8\pi\lambda^{3/4}}\hat{f}\left((-1)^j\frac{\lambda^{1/4}}{2\pi}\right)\overline{\hat{g}\left((-1)^j\frac{\lambda^{1/4}}{2\pi}\right)} \qquad \text{for almost every} \qquad \lambda \in \mathrm{Sp}(A) = [0,\infty).$$

Exercise 4.10

Let $f : \mathbb{R} \to \mathbb{C}$ be a bounded continuous function. Then

$$\int_{\mathbb{R}} f(x)\mu_v^\epsilon(x)\,\mathrm{d}x = \frac{1}{\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\epsilon f(x)}{\epsilon^2+(x-y)^2}\,\mathrm{d}\mu_v(y)\,\mathrm{d}x = \frac{1}{\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\epsilon f(x)}{\epsilon^2+(x-y)^2}\,\mathrm{d}x\,\mathrm{d}\mu_v(y),$$

where the second equality follows from Fubini's theorem. It is not immediately clear that we can apply the dominated convergence theorem. We first make a substitution of $\epsilon \tan u = x - y$ (that is, $u = \arctan(\frac{x-y}{\epsilon})$). Then we have:

$$\int_{\mathbb{R}} \frac{\epsilon f(x)}{\epsilon^2+(x-y)^2}\,\mathrm{d}x = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(y+\epsilon\tan u)\,\mathrm{d}u.$$

Taking $\epsilon \downarrow 0$, using the dominated convergence theorem and the continuity of f we obtain:

$$\int_{\mathbb{R}} \frac{\epsilon f(x)}{\epsilon^2+(x-y)^2}\,\mathrm{d}x \to \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(y)\,\mathrm{d}u = \pi f(y).$$

Since

$$\left|\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(y+\epsilon\tan u)\,\mathrm{d}u\right| \le \pi\|f\|_\infty$$

we can apply dominated convergence theorem once more to obtain

$$\lim_{\epsilon\downarrow 0}\frac{1}{\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\epsilon f(x)}{\epsilon^2+(x-y)^2}\,\mathrm{d}x\,\mathrm{d}\mu_v(y) = \int_{\mathbb{R}} f(y)\,\mathrm{d}\mu_v(y).$$

Since f was arbitrary, μ_v^ϵ converges weakly to μ_v as $\epsilon \downarrow 0$.

Exercise 4.11

The first part follows from the proof of Theorem 4.4.4. For the second part (optimality of the convergence bound), consider the case that $\rho_v(y) = |y|$ for $|y| \le 1$ and $\rho_v(y) = 0$ for $|y| > 1$. Then

$$\mu_v^\epsilon(0) = \frac{1}{\pi}\int_{-1}^{1} \frac{\epsilon|y|}{\epsilon^2+y^2}\,\mathrm{d}y = \frac{2\epsilon}{\pi}\int_0^{1/\epsilon} \frac{t}{1+t^2}\,\mathrm{d}t = \frac{\epsilon}{\pi}\log(1+1/\epsilon^2).$$

This integral is bounded below by a constant multiplied by $\epsilon\log(1/\epsilon)$ as $\epsilon \downarrow 0$. This shows the optimality. For the final part, assume that $\rho_v \in C^\infty(\mathcal{I})$. We follow the proof of Theorem 4.3.15. However, we now write $\mathcal{I}_0 = [-\eta, \eta]$ and

$$\begin{aligned}\rho_v(x_0) - \mu_v^\epsilon(x_0) &= \frac{1}{\pi}\int_{\mathcal{I}_0} \frac{\epsilon\rho_1(x_0)}{\epsilon^2+y^2}\,\mathrm{d}y + \frac{1}{\pi}\int_{\mathbb{R}\setminus\mathcal{I}_0} \frac{\epsilon\rho_1(x_0)}{\epsilon^2+y^2}\,\mathrm{d}y - \int_{\mathbb{R}} \frac{\epsilon\,\mathrm{d}\mu_v(y)}{\epsilon^2+(x_0-y)^2}\\ &= \frac{\epsilon}{\pi}\int_{\mathcal{I}_0} \frac{\rho_1(x_0)-\rho_1(x_0-y)}{\epsilon^2+y^2}\,\mathrm{d}y + \frac{\epsilon}{\pi}\int_{\mathbb{R}\setminus\mathcal{I}_0} \frac{\rho_1(x_0)-\rho_1(x_0-y)}{\epsilon^2+y^2}\,\mathrm{d}y - \int_{\mathbb{R}} \frac{\epsilon\,\mathrm{d}\mu_v^{(\mathrm{r})}(y)}{\epsilon^2+(x_0-y)^2}.\end{aligned}$$

We can show that the size of the second and third integrals on the right-hand side are $O(\epsilon)$ as before. For the first integral, we have

$$\rho_1(x_0) - \rho_1(x_0-y) = \rho_1'(x_0)y - \frac{1}{2}\rho_1''(\xi_y)y^2$$

for some ξ_y with $|\xi_y - x_0| \le y$. Due to the fact that the Poisson kernel is even, we have

$$\left|\frac{\epsilon}{\pi}\int_{\mathcal{I}_0} \frac{\rho_1(x_0)-\rho_1(x_0-y)}{\epsilon^2+y^2}\,\mathrm{d}y\right| \le \frac{\|\rho_1''\|_\infty}{2\pi}\int_{-\eta}^{\eta} \frac{\epsilon y^2}{\epsilon^2+y^2}\,\mathrm{d}y.$$

This final integral is $O(\epsilon)$. To see why this is sharp, we can simply take a constant function on $\mathcal{I}$ and bound the tail of the integral of the Poisson kernel.

Exercise 4.12

We prove the result by induction. Suppose that it is true for $\mu_\nu^{\epsilon,k}$. We let $\{a'_j\}_{j=1}^k$ and $\{\alpha'_j\}_{j=1}^k$ be the corresponding poles and coefficients (using the notation of Chapter 4). Let $\{a_j\}_{j=1}^{k+1}$ and $\{\alpha_j\}_{j=1}^{k+1}$ be the corresponding poles and coefficients for $\mu_\nu^{\epsilon,k+1}$. We may order these so that

$$\begin{aligned}
a_1 &= \frac{\mathrm{i}}{c^k}, & \alpha_1 &= \frac{c^k}{c^k-1}\alpha'_1,\\
a_2 &= \frac{\mathrm{i}}{c^{k-1}} = a'_1, & \alpha_2 &= \frac{c^k}{c^k-1}\alpha'_2 - \frac{1}{c^k-1}\alpha'_1,\\
&\vdots & &\vdots\\
a_k &= a'_{k-1}, & \alpha_k &= \frac{c^k}{c^k-1}\alpha'_k - \frac{1}{c^k-1}\alpha'_{k-1},\\
a_{k+1} &= a'_k, & \alpha_{k+1} &= -\frac{1}{c^k-1}\alpha'_k.
\end{aligned}$$

Since $\sum_{j=1}^k \alpha'_j = 1$, we have

$$\sum_{j=1}^{k+1}\alpha_j = \frac{c^k}{c^k-1} - \frac{1}{c^k-1} = 1.$$

In a similar fashion, let $l \in \mathbb{N}$ with $l \le k$. Then

$$\sum_{j=1}^{k+1} a_j^l \alpha_j = \sum_{j=1}^{k} a_j^l \frac{c^k}{c^k-1}\alpha'_j - \sum_{j=1}^{k}[a'_j]^l \frac{1}{c^k-1}\alpha'_j.$$

We note that $a_j^l = [a'_j]^l/c^l$ for $j = 1,\dots,k$. It follows that

$$\sum_{j=1}^{k+1} a_j^l \alpha_j = \left[\frac{c^{k-l}}{c^k-1} - \frac{1}{c^k-1}\right]\sum_{j=1}^{k}[a'_j]^l \alpha'_j.$$

If $l < k$, then the sum on the right-hand side is zero by our inductive hypothesis. If $l = k$, then the term in square brackets vanishes. We can perform a similar inductive argument for the poles in the lower half-plane and their coefficients. It follows that the Vandermonde system required for a high-order rational kernel is satisfied.

Exercise 4.13

The result boils down to showing that $|K(x)| = O(|x|^{-(m+2)})$ as $|x| \to \infty$ since then all the logarithmic error terms vanish in the analysis. From the analysis in Section 4.4.2 of the book, we see that this holds if

$$\sum_{j=1}^{m}\alpha_j a_j^m = \sum_{j=1}^{m}\beta_j b_j^m.$$

Since $\alpha_j = \overline{\beta_j}$ and $a_j = \overline{b_j}$, we are done if we can show that $\sum_{j=1}^m \alpha_j a_j^m$ is real. By construction, we have

$$\alpha_{m+1-j} = \overline{\alpha_j}, \quad a_{m+1-j} = -\overline{a_j}.$$

Hence,

$$\sum_{j=1}^{m/2}\alpha_j a_j^m = \sum_{j=1}^{m/2}\overline{\alpha_{m+1-j}}(-\overline{a_{m+1-j}})^m = \sum_{j=1}^{m/2}\overline{\alpha_{m+1-j}}\overline{a_{m+1-j}^m},$$

where the final equality holds since m is even. It follows that $\sum_{j=1}^m \alpha_j a_j^m$ is real.

Exercise 4.14

Applying Fubini's theorem, we see that

$$\int_a^b [K_\epsilon * \mathcal{E}](x)\,\mathrm{d}x = \frac{-1}{2\pi\mathrm{i}} \int_{\mathrm{Sp}(A)} \int_a^b \sum_{j=1}^m \left[\frac{\alpha_j}{\lambda - (x - \epsilon a_j)} - \frac{\bar{\alpha}_j}{\lambda - (x - \epsilon \bar{a}_j)}\right] \mathrm{d}x\, \mathrm{d}\mathcal{E}(\lambda).$$

To establish the theorem, we take the limit $\epsilon \to 0$ and apply the dominated convergence theorem to interchange the limit and the outer integral. This is permissible due to the decay condition of the kernel. We claim that, as $\epsilon \to 0$, the inner integral converges to $-2\pi\mathrm{i}$ when $\lambda \in (a, b)$, 0 when $\lambda \notin [a, b]$, and $(-2\pi\mathrm{i})c_l$ or $-(2\pi\mathrm{i})c_r$ when $\lambda = a$ or $\lambda = b$, respectively. We compute the inner integral directly by integrating the sum term by term, so that

$$\int_a^b \sum_{j=1}^m \left[\frac{\alpha_j}{\lambda - (x - \epsilon a_j)} - \frac{\bar{\alpha}_j}{\lambda - (x - \epsilon \bar{a}_j)}\right] \mathrm{d}x = \sum_{j=1}^m \left[\bar{\alpha}_j \log\left(\lambda - (x - \epsilon \bar{a}_j)\right) - \alpha_j \log\left(\lambda - (x - \epsilon a_j)\right)\right]\Big|_a^b.$$

Using the identity $\log(z) = \log|z| + \mathrm{i}\arg(z)$ to simplify, we find that the right-hand side is equal to

$$2\sum_{j=1}^m \left(\mathrm{Im}(\alpha_j)\left[\log|\lambda - b + \epsilon a_j| - \log|\lambda - a + \epsilon a_j|\right] - \mathrm{i}\mathrm{Re}(\alpha_j)\left[\arg(\lambda - b + \epsilon a_j) - \arg(\lambda - a + \epsilon a_j)\right]\right). \tag{11}$$

To calculate the limit, note that $\sum_{j=1}^m \alpha_j = 1$. In particular, $\sum_{j=1}^m \mathrm{Re}(\alpha_j) = 1$ and $\sum_{j=1}^m \mathrm{Im}(\alpha_j) = 0$. Then, the right-hand terms involving arg evaluate to

$$\lim_{\epsilon\to 0} \sum_{j=1}^m \mathrm{Re}(\alpha_j)\left[\arg(\lambda - b + \epsilon a_j) - \arg(\lambda - a + \epsilon a_j)\right] = \begin{cases} \pi, & a < \lambda < b, \\ \sum_{j=1}^m \mathrm{Re}(\alpha_j)(\pi - \arg(a_j)), & \lambda = a, \\ \sum_{j=1}^m \mathrm{Re}(\alpha_j)\arg(a_j), & \lambda = b, \\ 0, & \text{otherwise.} \end{cases} \tag{12}$$

On the other hand, the left-hand terms involving logarithms vanish when $\lambda \neq a$ and $\lambda \neq b$, that is,

$$\lim_{\epsilon\to 0} \sum_{j=1}^m \mathrm{Im}(\alpha_j)\left[\log|\lambda - b + \epsilon a_j| - \log|\lambda - a + \epsilon a_j|\right] = \left[\log|\lambda - b| - \log|\lambda - a|\right] \sum_{j=1}^m \mathrm{Im}(\alpha_j) = 0. \tag{13}$$

Finally, when $\lambda = b$ we expand $\log|\epsilon a_j| = \log|\epsilon| + \log|a_j|$ and perform a similar calculation to obtain

$$\lim_{\epsilon\to 0} \sum_{j=1}^m \mathrm{Im}(\alpha_j)\left[\log|\epsilon a_j| - \log|b - a + \epsilon a_j|\right] = \sum_{j=1}^m \mathrm{Im}(\alpha_j)\log|a_j|. \tag{14}$$

We omit the analogous calculation for $\lambda = a$, which only differs by a minus sign. Collecting the results in (11), (12), (13) and (14) establishes the first part of the proposition.

For the second part of the proposition, suppose that the poles are symmetric about the imaginary axis. This symmetry implies that $\mathrm{Im}(\alpha_{m+1-j}) = -\mathrm{Im}(\alpha_j)$ and $\log|a_j| = \log|a_{m+1-j}|$. Therefore, the logarithmic terms in c_l and c_r vanish because

$$\sum_{j=1}^m \mathrm{Im}(\alpha_j)\log|a_j| = \sum_{j=1}^{\lfloor m/2 \rfloor} (\mathrm{Im}(\alpha_j)\log|a_j| + \mathrm{Im}(\alpha_{m+1-j})\log|a_{m+1-j}|) = 0.$$

Furthermore, the pole symmetries $a_{m+1-j} = -\bar{a}_j$ imply that $\arg(a_{m+1-j}) = \pi - \arg(a_j)$, while the residue symmetries also imply that $\mathrm{Re}(\alpha_{m+1-j}) = \mathrm{Re}(\alpha_j)$. Therefore, we find that

$$c_l = \pi^{-1} \sum_{j=1}^m \mathrm{Re}(\alpha_j)\left(\pi - \arg(a_j)\right) = \pi^{-1} \sum_{j=1}^m \mathrm{Re}(\alpha_j)\arg(a_j) = c_r. \tag{15}$$

Now, observe that $\mathrm{Re}(\alpha_j)\arg(a_j) + \mathrm{Re}(\alpha_{m+1-j})\arg(a_{m+1-j}) = \pi\mathrm{Re}(\alpha_j)$. For even m, we calculate that

$$\sum_{j=1}^m \mathrm{Re}(\alpha_j)\arg(a_j) = \sum_{j=1}^{m/2} \left(\mathrm{Re}(\alpha_j)\arg(a_j) + \mathrm{Re}(\alpha_{m+1-j})\arg(a_{m+1-j})\right) = \pi \sum_{j=1}^{m/2} \mathrm{Re}(\alpha_j) = \frac{\pi}{2}. \tag{16}$$

The last equality follows from the fact that $2\sum_{j=1}^{m/2}\mathrm{Re}(\alpha_j) = \sum_{j=1}^{m}\mathrm{Re}(\alpha_j) = 1$ when m is even. Analogously for odd m, we obtain

$$\sum_{j=1}^{m}\mathrm{Re}(\alpha_j)\arg(a_j) = \frac{\pi}{2}\mathrm{Re}(\alpha_{\lceil m/2\rceil}) + \pi\sum_{j=1}^{\lfloor m/2\rfloor}\mathrm{Re}(\alpha_j) = \frac{\pi}{2}. \tag{17}$$

Here, we have used that $\mathrm{Re}(\alpha_{\lceil m/2\rceil}) + 2\sum_{j=1}^{\lfloor m/2\rfloor}\mathrm{Re}(\alpha_j) = \sum_{j=1}^{m}\mathrm{Re}(\alpha_j) = 1$ when m is odd. Plugging (16) and (17) into (15) demonstrates that $c_l = c_r = 1/2$, which concludes the proof.

Exercise 4.15

Let $x \in \mathbb{R}$ be closer to the λ than the rest of the spectrum. We first write

$$\upsilon_v^{\epsilon}(x) = \frac{\epsilon^2\langle\mathcal{P}_\lambda v, v\rangle}{\epsilon^2 + (x-\lambda)^2} + \int_{\mathrm{Sp}(A)\setminus\{\lambda\}} \frac{\epsilon^2\,\mathrm{d}\mu_v(y)}{\epsilon^2 + (x-y)^2}.$$

When we differentiate with respect to x, the derivative of the second term is $O(\epsilon^2)$ as λ is separated from the rest of the spectrum. Differentiating the first term gives the required result.

Exercise 4.16

This is self-explanatory and no answer is needed. The code for the examples in the chapter can be found in the repository.

5 Chapter 5

Exercise 5.1

Let

$$v = \sum_{j=-N}^{N} v_j e_j, \quad w = \sum_{j=-N}^{N} w_j e_j.$$

To compute the spectral measure $\xi_{v,w}$, we compute Fourier coefficients

$$\widehat{\xi_{v,w}}(n) = \frac{1}{2\pi}\int_{[-\pi,\pi]_{\mathrm{per}}} e^{-\mathrm{i}n\theta}\,\mathrm{d}\xi_{v,w}(\theta) = \frac{1}{2\pi}\langle v, A^n w\rangle = \frac{1}{2\pi}\sum_{j=-N}^{N}\sum_{k=-N}^{N} v_j\overline{w_k}\langle e_j, e_{k-n}\rangle = \frac{1}{2\pi}\sum_{k=-N}^{N} v_{k-n}\overline{w_k},$$

where we set $v_m = 0$ if $|m| > N$. It is clear that these coefficients vanish for sufficiently large $|n|$. Hence, $\xi_{v,w}$ is absolutely continuous and its Radon–Nikodym derivative is the trigonometric polynomial

$$\sum_{|n|\leq 2N}\left[\frac{1}{2\pi}\sum_{k=-N}^{N} v_{k-n}\overline{w_k}\right]e^{in\theta}.$$

Exercise 5.2

Let $A \in \Omega_{\mathrm{U}}$ be unitary. Then $A^*A = AA^* = I$ and, hence, we have

$$(A - zI)^*(A - zI) = (1 + |z|^2)I - \bar{z}A - zA^*, \quad (A - zI)(A - zI)^* = (1 + |z|^2)I - \bar{z}A - zA^*.$$

It follows that operator folding can be performed given access to only the matrix elements of A. (Given $A \in \Omega_{\mathrm{U}}$, we can also evaluate, using Λ, an A-dependent function $f : \mathbb{N} \to \mathbb{N}$ pointwise such that $A \in \Omega_f$.) The techniques of Chapter 3 can be applied (a unitary operator is normal so we can take $g_m(x) = x$) to see that $\{\mathrm{Sp}, \Omega_{\mathrm{U}}, \mathcal{M}_{\mathrm{H}}, \Lambda\} \in \Sigma_1^A$.

Exercise 5.3

Consider $\mathcal{H} = \ell^2(\mathbb{N})$ and let A be defined by $Ae_j = e_{j+1}$. Then A is an isometry but, $AA^*e_1 = 0$, so A is not unitary.

Now suppose that A is an isometry but not unitary. Then $0 \in \mathrm{Sp}(A)$ and $\mathrm{Sp}(A) \subset \{z \in \mathbb{C} : |z| \leq 1\}$. Suppose for a contradiction that $\mathrm{Sp}(A) \neq \{z \in \mathbb{C} : |z| \leq 1\}$. Then there must exist some $\lambda \in \partial\mathrm{Sp}(A) \subset \mathrm{Sp}_{\mathrm{ap}}(A)$ with $|\lambda| < 1$. In particular, $\sigma_{\mathrm{inf}}(A - \lambda I) = 0$. But for every vector x with $\|x\| = 1$, we have

$$\|(A - \lambda I)x\| \geq \|Ax\| - |\lambda| = \|x\| - |\lambda| = 1 - |\lambda|,$$

which is a contradiction.

Suppose now that $A \in \Omega_{\mathrm{iso}}$. If A is unitary, $\sigma_{\mathrm{inf}}(A^*) = 1$, otherwise $\sigma_{\mathrm{inf}}(A^*) = 0$. There exists an arithmetic tower $\{\Gamma^0_{n_2,n_1}\}$ using Λ such that

$$\lim_{n_2\to\infty}\lim_{n_1\to\infty}\Gamma^0_{n_2,n_1}(A) = \sigma_{\mathrm{inf}}(A^*).$$

Moreover, we can ensure that the final limit is from above and that the first limit is eventually constant for each fixed n_2. This uses the machinery of rectangular truncations in Chapter 3. Let $\{\Gamma_n^U\}$ be the Σ_1^A tower for $\{\mathrm{Sp}, \Omega_{\mathrm{U}}, \mathcal{M}_{\mathrm{H}}, \Lambda\}$ from the previous exercise. We then set, for $A \in \Omega_{\mathrm{iso}}$,

$$\Gamma_{n_2,n_1}(A) = \begin{cases}\Gamma^U_{n_2}(A), & \text{if } \Gamma^0_{n_2,n_1}(A) > 1/2,\\ \{z \in \mathbb{C} : |z| \leq 1\}, & \text{otherwise.}\end{cases}$$

The fact that $\Gamma^0_{n_2,n_1}(A)$ is constant for large n_1 (for each fixed n_2) ensures that

$$\lim_{n_1\to\infty}\Gamma_{n_2,n_1}(A) = \begin{cases}\Gamma^U_{n_2}(A), & \text{if } \Gamma^0_{n_2}(A) > 1/2,\\ \{z \in \mathbb{C} : |z| \leq 1\}, & \text{otherwise.}\end{cases}$$

Hence, $\{\Gamma_{n_2,n_1}\}$ provides a Σ_2^A-tower for $\{\mathrm{Sp}, \Omega_{\mathrm{iso}}, \mathcal{M}_{\mathrm{H}}, \Lambda\}$.

Suppose for a contradiction that $\{\Gamma_n\}$ is a Δ_2^G-tower for $\{\mathrm{Sp}, \Omega_{\mathrm{iso}}, \mathcal{M}_{\mathrm{H}}, \Lambda\}$. We let $\Omega' = \{0,1\}^{\mathbb{N}}$ and let Λ' be the set of component-wise evaluations. Let $\Xi(a) = 1$ if $a = \{a_j\}_{j=1}^{\infty} \in \Omega'$ has infinitely many 1's and $\Xi(a) = 0$ otherwise. We have $\{\Xi, \Omega', [0,1], \Lambda'\} \notin \Delta_2^G$. Given $a \in \Omega'$, we construct a matrix

$$A(a) = \bigoplus_{j=1}^{\infty} C_{l_j}, \quad C_{l_j} = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ 1 & & & 0 \end{pmatrix} \in \mathbb{R}^{l_j \times l_j},$$

where C_∞ is taken to be the corresponding unilateral shift and we choose l_j to be double the gap (in indices) between the jth and $(j+1)$th 1's in the sequence a (where we extend to $a_0 = 1$), and take $l_j = \infty$ for $j \geq k$ if there are only k 1's in the sequence a. In this case, $A(a) \in \Omega_{\mathrm{iso}}$ and is unitary if and only if $\Xi(a) = 1$. We can now construct the embedding and contradiction by setting

$$\Gamma_n'(a) = 1 - \max\{1 - \mathrm{dist}(\Gamma_n(A(a)), 0), 0\}$$

and arguing in the usual way.

Finally, for the class $\Omega_{\mathrm{iso}} \cap \Omega_f$, we can replace $\{\Gamma_{n_2,n_1}^0\}$ by a corresponding Π_1^A-tower by taking $f(n_1) \times n_1$ truncations of matrices. This adaptation yields a Σ_1^A-tower.

Exercise 5.4

We first note that, by Fubini's theorem and using a tangent half-angle substitution,

$$\frac{1}{4\pi} \int_a^b F_{\mathcal{E}}((1+\epsilon)^{-1} e^{i\theta}) - F_{\mathcal{E}}((1+\epsilon)e^{i\theta}) \, d\theta = \int_{[-\pi,\pi]_{\mathrm{per}}} \frac{1}{\pi} \left[\tan^{-1}\left(\frac{(r+1)\tan\left(\frac{b-\varphi}{2}\right)}{1-r} \right) - \tan^{-1}\left(\frac{(r+1)\tan\left(\frac{a-\varphi}{2}\right)}{1-r} \right) \right] d\mathcal{E}(\varphi),$$

where $r = (1+\epsilon)^{-1}$. The proof now follows the self-adjoint case (see the proof of Theorem 4.3.1).

Exercise 5.5

Let f be a continuous function on $[-\pi,\pi]_{\mathrm{per}}$ and $r = (1+\epsilon)^{-1}$. Then, by Fubini's theorem,

$$\int_{[-\pi,\pi]_{\mathrm{per}}} f(\theta) \xi_v^{\epsilon}(\theta) \, d\theta = \frac{1}{2\pi} \int_{[-\pi,\pi]_{\mathrm{per}}} \int_{[-\pi,\pi]_{\mathrm{per}}} \frac{(1-r^2) f(\theta)}{1 + r^2 - 2r\cos(\theta - \varphi)} \, d\theta \, d\xi_v(\varphi).$$

We now make the substitution (motivated by Exercise 5.4) $u = 2\tan^{-1}(\frac{r+1}{1-r}\tan(\frac{\theta-\varphi}{2}))$ to see that

$$\int_{[-\pi,\pi]_{\mathrm{per}}} \frac{(1-r^2) f(\theta)}{1 + r^2 - 2r\cos(\theta - \varphi)} \, d\theta = \int_{-\pi}^{\pi} f\left(\varphi + 2\tan^{-1}\left(\frac{1-r}{1+r} \tan\left(\frac{u}{2}\right)\right)\right) du.$$

We may use the continuity of f and the dominated convergence theorem to see that this integral converges to $2\pi f(\varphi)$ as $\epsilon \downarrow 0$. The proof now follows the self-adjoint case.

Exercise 5.6

The proofs go through by using convolutions with the Poisson kernel for the unit disc instead of the half-plane.

Exercise 5.7

Proof of Theorem 5.4.9. Taking $\mathcal{P}_{e^{i\theta_0}} = 0$ in the case that $e^{i\theta_0}$ is not an eigenvalue,

$$\frac{1}{K_\epsilon^{\mathbb{T}}(0)} [K_\epsilon^{\mathbb{T}} * \mathcal{E}](\theta_0) \mathcal{P}_{e^{i\theta_0}} v = \frac{1}{K_\epsilon^{\mathbb{T}}(0)} \int_{[-\pi,\pi]_{\mathrm{per}}} K_\epsilon^{\mathbb{T}}(\theta_0 - \varphi) \, d\mathcal{E}(\varphi) \mathcal{P}_{e^{i\theta_0}} v = \mathcal{P}_{e^{i\theta_0}} v.$$

It follows that we can define

$$v'_\epsilon = \frac{1}{K_\epsilon^{\mathbb{T}}(0)}[K_\epsilon^{\mathbb{T}} * \mathcal{E}](\theta_0)v - \mathcal{P}_{e^{\mathrm{i}\theta_0}}v = \frac{1}{K_\epsilon^{\mathbb{T}}(0)}[K_\epsilon^{\mathbb{T}} * \mathcal{E}](\theta_0)(v - \mathcal{P}_{e^{\mathrm{i}\theta_0}}v).$$

From the functional calculus, we have

$$\|v'_\epsilon\|^2 = \int_{[-\pi,\pi]_{\mathrm{per}}} \frac{|K_\epsilon^{\mathbb{T}}(\theta_0-\varphi)|^2}{|K_\epsilon^{\mathbb{T}}(0)|^2}\,\mathrm{d}\xi_{v-\mathcal{P}_{e^{\mathrm{i}\theta_0}}v}(\varphi). \tag{18}$$

The concentration bound in the definition of a high-order periodic kernel and the condition $\liminf_{\epsilon\downarrow 0}\epsilon|K_\epsilon^{\mathbb{T}}(0)| > 0$ imply that

$$\frac{|K_\epsilon^{\mathbb{T}}(\theta_0-\varphi)|^2}{|K_\epsilon^{\mathbb{T}}(0)|^2} \lesssim \frac{1}{\epsilon^2|K_\epsilon^{\mathbb{T}}(0)|^2}\frac{1}{(1+|\theta_0-\varphi|/\epsilon)^{2m+2}}$$

is uniformly bounded and converges to zero as $\epsilon \downarrow 0$ whenever $\varphi \neq \theta_0$. Since $\xi_{v-\mathcal{P}_{e^{\mathrm{i}\theta_0}}v}(\{\theta_0\}) = 0$, we can apply the dominated convergence theorem to the right-hand side of (18) to see that $\lim_{\epsilon\downarrow 0}\|v'_\epsilon\|^2 = 0$, which proves the result. □

Proof of Theorem 5.4.10. Using the functional calculus, we have

$$(A - e^{\mathrm{i}\theta_0}I)u_\epsilon = \left[\int_{[-\pi,\pi]_{\mathrm{per}}} (e^{\mathrm{i}\varphi} - e^{\mathrm{i}\theta_0})K_\epsilon^{\mathbb{T}}(\theta_0-\varphi)\,\mathrm{d}\mathcal{E}(\varphi)\right]v.$$

This implies that

$$\|(A - e^{\mathrm{i}\theta_0}I)u_\epsilon\|^2 = \int_{[-\pi,\pi]_{\mathrm{per}}} |e^{\mathrm{i}\varphi} - e^{\mathrm{i}\theta_0}|^2|K_\epsilon^{\mathbb{T}}(\theta_0-\varphi)|^2\,\mathrm{d}\xi_v(\varphi).$$

Similarly, we have

$$\|u_\epsilon\|^2 = \int_{[-\pi,\pi]_{\mathrm{per}}} |K_\epsilon^{\mathbb{T}}(\theta_0-\varphi)|^2\,\mathrm{d}\xi_v(\varphi).$$

Suppose that $\epsilon\eta < \delta$ and let

$$I_\delta = \int_{|\theta_0-\varphi|\leq\delta} |K_\epsilon^{\mathbb{T}}(\theta_0-\varphi)|^2\,\mathrm{d}\xi_v(\varphi) \geq \left(\inf_{|\theta|<\eta\epsilon}\epsilon|K_\epsilon^{\mathbb{T}}(\theta)|\right)^2\frac{1}{\epsilon^2}\int_{|\theta_0-\varphi|\leq\eta\epsilon} 1\,\mathrm{d}\xi_v(\varphi).$$

The two bounds in the statement of the theorem imply that $\liminf_{\epsilon\downarrow 0} I_\delta > 0$. We have

$$\begin{aligned}\int_{[-\pi,\pi]_{\mathrm{per}}} |e^{\mathrm{i}\varphi} - e^{\mathrm{i}\theta_0}|^2|K_\epsilon^{\mathbb{T}}(\theta_0-\varphi)|^2\,\mathrm{d}\xi_v(\varphi) &\leq \left(\sup_{|\theta|\leq\delta}|1-e^{\mathrm{i}\theta}|^2\right)I_\delta + \int_{|\theta_0-\varphi|>\delta} |e^{\mathrm{i}\varphi} - e^{\mathrm{i}\theta_0}|^2|K_\epsilon^{\mathbb{T}}(\theta_0-\varphi)|^2\,\mathrm{d}\xi_v(\varphi)\\ &\lesssim \delta^2 I_\delta + \int_{|\theta_0-\varphi|>\delta} |e^{\mathrm{i}\varphi} - e^{\mathrm{i}\theta_0}|^2|K_\epsilon^{\mathbb{T}}(\theta_0-\varphi)|^2\,\mathrm{d}\xi_v(\varphi).\end{aligned}$$

Using the concentration condition in the definition of a high-order periodic kernel, we see that

$$\lim_{\epsilon\downarrow 0}\int_{|\theta_0-\varphi|>\delta} |e^{\mathrm{i}\varphi} - e^{\mathrm{i}\theta_0}|^2|K_\epsilon^{\mathbb{T}}(\theta_0-\varphi)|^2\,\mathrm{d}\xi_v(\varphi) = 0.$$

Similarly, we have

$$\int_{[-\pi,\pi]_{\mathrm{per}}} |K_\epsilon^{\mathbb{T}}(\theta_0-\varphi)|^2\,\mathrm{d}\xi_v(\varphi) \geq I_\delta - \int_{|\theta_0-\varphi|>\delta} |K_\epsilon^{\mathbb{T}}(\theta_0-\varphi)|^2\,\mathrm{d}\xi_v(\varphi)$$

and

$$\lim_{\epsilon\downarrow 0}\int_{|\theta_0-\varphi|>\delta} |K_\epsilon^{\mathbb{T}}(\theta_0-\varphi)|^2\,\mathrm{d}\xi_v(\varphi) = 0.$$

Hence,

$$\limsup_{\epsilon\downarrow 0}\frac{\|(A - e^{\mathrm{i}\theta_0}I)u_\epsilon\|^2}{\|u_\epsilon\|^2} \lesssim \limsup_{\epsilon\downarrow 0}\frac{\delta^2 I_\delta + \int_{|\theta_0-\varphi|>\delta}|e^{\mathrm{i}\varphi} - e^{\mathrm{i}\theta_0}|^2|K_\epsilon^{\mathbb{T}}(\theta_0-\varphi)|^2\,\mathrm{d}\xi_v(\varphi)}{I_\delta - \int_{|\theta_0-\varphi|>\delta}|K_\epsilon^{\mathbb{T}}(\theta_0-\varphi)|^2\,\mathrm{d}\xi_v(\varphi)} = \delta^2.$$

Since $\delta > 0$ was arbitrary, the result now follows. □

Exercise 5.8

Throughout this solution, we use the substitution $x = \tan(\theta/2)$. Using this substitution, we see that

$$\int_{-\pi}^{\pi} K_\epsilon^{\mathbb{T}}(\theta)\,\mathrm{d}\theta = \int_{-\infty}^{\infty} K_\epsilon(x)\,\mathrm{d}x = 1.$$

We have

$$\exp(-2\mathrm{i}\tan^{-1}(x)) = \frac{1-\mathrm{i}x}{1+\mathrm{i}x},$$

and hence

$$\exp(-2\mathrm{i}n\tan^{-1}(x)) - 1 = \left(\frac{1-\mathrm{i}x}{1+\mathrm{i}x}\right)^n - 1.$$

It follows that

$$\int_{-\pi}^{\pi} K_\epsilon^{\mathbb{T}}(\theta)e^{-\mathrm{i}n\theta}\,\mathrm{d}\theta - 1 = \int_{-\infty}^{\infty} K_\epsilon(x)[\exp(-2\mathrm{i}n\tan^{-1}(x)) - 1]\,\mathrm{d}x = \int_{-\infty}^{\infty} K_\epsilon(x)\left[\left(\frac{1-\mathrm{i}x}{1+\mathrm{i}x}\right)^n - 1\right]\mathrm{d}x = [K_\epsilon * \phi](0),$$

where (taking care of the argument $-x$ in the convolution)

$$\phi(x) = \left(\frac{1+\mathrm{i}x}{1-\mathrm{i}x}\right)^n - 1.$$

Since ϕ is smooth and with $\phi(0) = 0$, the pointwise convergence rate of mth order kernels for $\mathbb{R}$ implies that

$$\left|\int_{-\pi}^{\pi} K_\epsilon^{\mathbb{T}}(\theta)e^{-\mathrm{i}n\theta}\,\mathrm{d}\theta - 1\right| = O(\epsilon^m \log(1+\epsilon^{-1})).$$

For the concentration inequality, we have from the decay condition of K that (for some constant C):

$$\left|K_\epsilon^{\mathbb{T}}(\theta)\right| \le \frac{1}{2\epsilon}\sec^2\left(\frac{\theta}{2}\right)\frac{C}{\left(1+\left|\tan\left(\frac{\theta}{2}\right)\right|/\epsilon\right)^{m+1}} = \frac{C}{2}\sec^2\left(\frac{\theta}{2}\right)\frac{\epsilon^m}{\left(\epsilon+\left|\tan\left(\frac{\theta}{2}\right)\right|\right)^{m+1}}.$$

For $|\theta| < 1$, this bound is bounded by $C_1\epsilon^m/(\epsilon+|\theta|)^{m+1}$ for some constant C_1. For $|\theta| \ge 1$, it is bounded by some constant C_2 (since $\epsilon \le 1$). It follows that

$$\left|K_\epsilon^{\mathbb{T}}(\theta)\right| \le \frac{C_3\epsilon^m}{(\epsilon+|\theta|)^{m+1}}$$

for some constant C_3 that is independent of ϵ and θ.

Now suppose that K is of the rational form in the last part of the exercise. Then

$$\begin{aligned}
K_\epsilon^{\mathbb{T}}(\theta) &= \frac{1}{2}\left[\left(\frac{\mathrm{i}(e^{-\mathrm{i}\theta}-1)}{(e^{-\mathrm{i}\theta}+1)}\right)^2 + 1\right]\frac{1}{2\pi\mathrm{i}}\sum_{j=1}^m\left(\frac{\alpha_j}{\frac{\mathrm{i}(e^{-\mathrm{i}\theta}-1)}{(e^{-\mathrm{i}\theta}+1)} - \epsilon a_j} - \frac{\beta_j}{\frac{\mathrm{i}(e^{-\mathrm{i}\theta}-1)}{(e^{-\mathrm{i}\theta}+1)} - \epsilon b_j}\right)\\
&= \frac{1}{2}\left[\left(\frac{\mathrm{i}(e^{-\mathrm{i}\theta}-1)}{(e^{-\mathrm{i}\theta}+1)}\right)^2/\epsilon + 1/\epsilon\right]\frac{1}{2\pi\mathrm{i}}\sum_{j=1}^m\left(\frac{\alpha_j}{\frac{\mathrm{i}(e^{-\mathrm{i}\theta}-1)}{(e^{-\mathrm{i}\theta}+1)}/\epsilon - a_j} - \frac{\beta_j}{\frac{\mathrm{i}(e^{-\mathrm{i}\theta}-1)}{(e^{-\mathrm{i}\theta}+1)}/\epsilon - b_j}\right)\\
&= \frac{1}{4\pi\mathrm{i}}\sum_{j=1}^m\left(\left[a_j\frac{\mathrm{i}(e^{-\mathrm{i}\theta}-1)}{(e^{-\mathrm{i}\theta}+1)} + 1/\epsilon\right]\frac{\alpha_j}{\frac{\mathrm{i}(e^{-\mathrm{i}\theta}-1)}{(e^{-\mathrm{i}\theta}+1)}/\epsilon - a_j} - \left[b_j\frac{\mathrm{i}(e^{-\mathrm{i}\theta}-1)}{(e^{-\mathrm{i}\theta}+1)} + 1/\epsilon\right]\frac{\beta_j}{\frac{\mathrm{i}(e^{-\mathrm{i}\theta}-1)}{(e^{-\mathrm{i}\theta}+1)}/\epsilon - b_j}\right),
\end{aligned}$$

where the final equality uses the equation referenced from Chapter 4. We then note that

$$\begin{aligned}
\frac{1}{4\pi\mathrm{i}}\left[a_j\frac{\mathrm{i}(e^{-\mathrm{i}\theta}-1)}{(e^{-\mathrm{i}\theta}+1)} + 1/\epsilon\right]\frac{\alpha_j}{\left(\frac{\mathrm{i}(e^{-\mathrm{i}\theta}-1)}{(e^{-\mathrm{i}\theta}+1)}\right)/\epsilon - a_j} &= \frac{1}{4\pi}\left[\epsilon a_j(e^{-\mathrm{i}\theta}-1) - \mathrm{i}(e^{-\mathrm{i}\theta}+1)\right]\frac{\alpha_j}{\mathrm{i}(e^{-\mathrm{i}\theta}-1) - \epsilon a_j(e^{-\mathrm{i}\theta}+1)}\\
&= \frac{-\alpha_j}{4\pi}\frac{\mathrm{i}(e^{-\mathrm{i}\theta}+1) - \epsilon a_j(e^{-\mathrm{i}\theta}-1)}{\mathrm{i}(e^{-\mathrm{i}\theta}-1) - \epsilon a_j(e^{-\mathrm{i}\theta}+1)}\\
&= \frac{-\alpha_j}{4\pi}\frac{e^{-\mathrm{i}\theta} + \frac{\mathrm{i}+\epsilon a_j}{\mathrm{i}-\epsilon a_j}}{e^{-\mathrm{i}\theta} - \frac{\mathrm{i}+\epsilon a_j}{\mathrm{i}-\epsilon a_j}}.
\end{aligned}$$

We can perform the same steps with b_j and β_j replacing a_j and α_j to see that the expression in the statement of the exercise holds. It follows that

$$\left[K_\epsilon^{\mathbb{T}} * \mathcal{E}\right](\theta) = \frac{-1}{4\pi}\sum_{j=1}^{m}\left(\alpha_j F_{\mathcal{E}}\left(e^{\mathrm{i}\theta}\frac{\mathrm{i}+\epsilon a_j}{\mathrm{i}-\epsilon a_j}\right) - \beta_j F_{\mathcal{E}}\left(e^{\mathrm{i}\theta}\frac{\mathrm{i}+\epsilon b_j}{\mathrm{i}-\epsilon b_j}\right)\right).$$

Exercise 5.9

We start with the periodic summation of the Poisson kernel (for the upper half-plane):

$$K_{\epsilon,1}^{\mathbb{T}}(\theta) = \sum_{n=-\infty}^{\infty}\frac{\epsilon}{\pi}\frac{1}{\epsilon^2+(\theta+2\pi n)^2} = \frac{1}{2\pi\mathrm{i}}\sum_{n=-\infty}^{\infty}\left(\frac{1}{\theta+2\pi n-\mathrm{i}\epsilon} - \frac{1}{\theta+2\pi n+\mathrm{i}\epsilon}\right) = \frac{1}{4\pi\mathrm{i}}\left[\cot\left(\frac{\theta-\mathrm{i}\epsilon}{2}\right) - \cot\left(\frac{\theta+\mathrm{i}\epsilon}{2}\right)\right].$$

We then define inductively:

$$K_{\epsilon,k+1}^{\mathbb{T}}(\theta) = \frac{c^k K_{\epsilon/c,k}^{\mathbb{T}}(\theta) - K_{\epsilon,k}^{\mathbb{T}}(\theta)}{c^k-1},$$

where $c > 1$. These kernels can be written explicitly in terms of the cot function. Periodic summation and this form of extrapolation commute, and, hence, we see that $K_{\epsilon,k}^{\mathbb{T}}$ is the periodic sum of the corresponding kth order kernel (for $\mathbb{R}$) in Exercise 4.12.

Exercise 5.10

Define the filter function

$$\sigma(x) = (1-|x|)\cos(\pi x) + \frac{\sin(\pi|x|)}{\pi},$$

then it is easily checked that this is a second-order filter. Letting $N = \lfloor \epsilon^{-1}\rfloor$, the kernel associated with the Jackson factors can be written as a sum of the second-order kernel generated by this filter and a remainder:

$$\begin{aligned}&\left(\frac{1}{2\pi}\sum_{n=-N}^{N}\sigma\left(\frac{n}{N}\right)e^{\mathrm{i}n\theta}\right) + \frac{1}{2\pi}\left(\frac{\cot(\pi/N)}{N} - \frac{1}{\pi}\right)\sum_{n=-N}^{N}\sin\left(\frac{|n|}{N}\pi\right)e^{\mathrm{i}n\theta}\\ &= \left(\frac{1}{2\pi}\sum_{n=-N}^{N}\sigma\left(\frac{n}{N}\right)e^{\mathrm{i}n\theta}\right) + \frac{1}{2\pi}\left(\frac{\cot(\pi/N)}{N} - \frac{1}{\pi}\right)\sin\left(\frac{\pi}{N}\right)\frac{\cos^2\left(\frac{N\theta}{2}\right)}{\sin\left(\frac{\pi+N\theta}{2N}\right)\sin\left(\frac{\pi-N\theta}{2N}\right)}.\end{aligned}$$

Let

$$h_N(\theta) = \frac{1}{2\pi}\left(\frac{\cot(\pi/N)}{N} - \frac{1}{\pi}\right)\sin\left(\frac{\pi}{N}\right)\frac{\cos^2\left(\frac{N\theta}{2}\right)}{\sin\left(\frac{\pi+N\theta}{2N}\right)\sin\left(\frac{\pi-N\theta}{2N}\right)}.$$

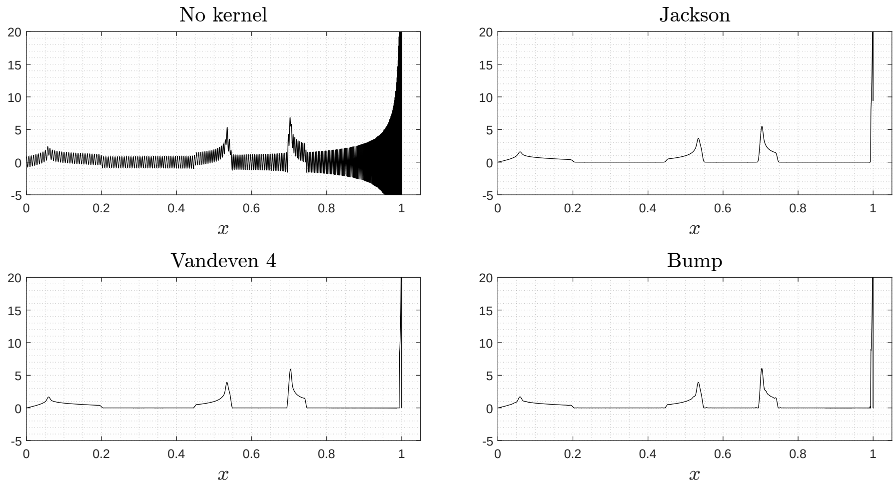
Then $\int_{-\pi}^{\pi} h_N(\theta)\,\mathrm{d}\theta = 0$. Moreover, as $N \to \infty$,

$$\left|\int_{-\pi}^{\pi} h_N(\theta)e^{-\mathrm{i}n\theta}\,\mathrm{d}\theta\right| = \left|\left(\frac{\cot(\pi/N)}{N} - \frac{1}{\pi}\right)\sin\left(\frac{|n|}{N}\pi\right)\right| = O(|n|N^{-3}).$$

Finally, we have

$$|h_N(\theta)| \le \frac{CN^{-2}}{(N^{-1}+|\theta|)^3},$$

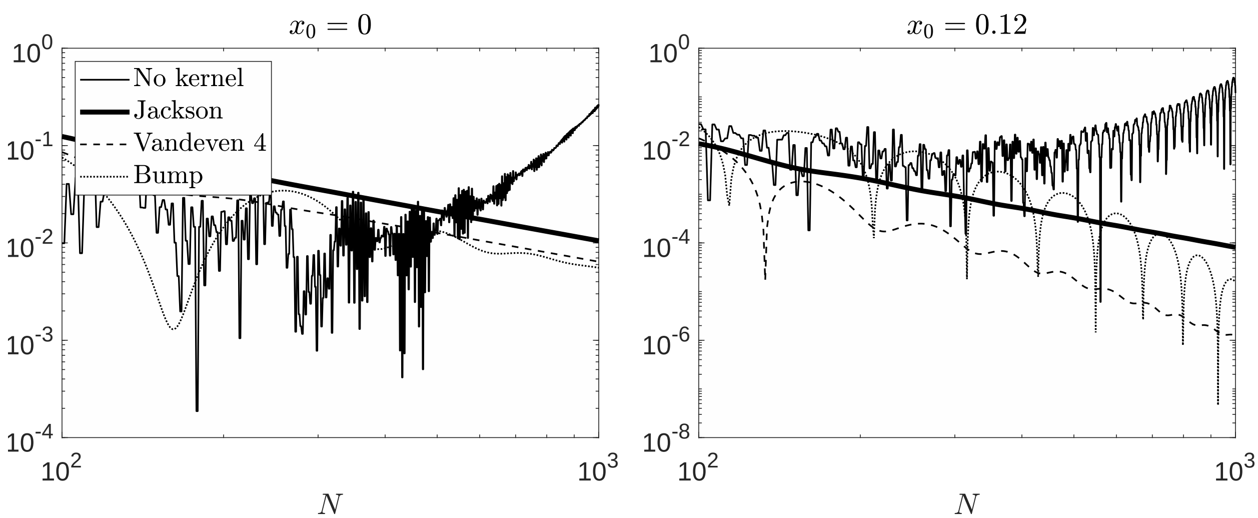
for some constant C. It follows that the kernel associated with the Jackson factors is a second-order kernel.



Exercise 5.11

Code for this exercise can be found in “ex5_11.m” in the repository We revisit the graphene operator and set $\Phi = 1/4$. We rescale the operator so its spectrum lies in $[-1, 1]$ by approximating the largest eigenvalue of a truncation. The following figure shows the approximations of μ_{e_1} (the Legendre series) with no filter and various filters at $N = 1000$. For the Vandeven 4 and bump filters, we have plotted the maximum of the approximations and zero. The severe Gibbs oscillations with no filter are clearly visible.

The figure also shows the pointwise errors at $x_0 = 0$ and $x_0 = 0.12$. The measure is not smooth at $x_0 = 0$, and all filtered methods converge at the same rate. In contrast, at $x_0 = 0.12$, the convergence rate follows the order of the filter. The sum with no filter diverges at both points as $N \to \infty$.



6 Chapter 6

Exercise 6.1

Suppose that A_j acts on the Hilbert space $\mathcal{H}_j$. For $\diamond \in \{\mathrm{ac}, \mathrm{sc}, \mathrm{pp}, \mathrm{c}, \mathrm{s}\}$, write $\mathcal{H}_{j,\diamond}$ and $\mathcal{H}_\diamond$ for the $\diamond$ components of $\mathcal{H}_j$ with respect to A_j and $\mathcal{H} = \oplus_{j=1}^\infty \mathcal{H}_j$ with respect to A, respectively. Let $\mathcal{E}_A$ be the projection-valued spectral measure associated with A and $\mathcal{E}_{A_j}$ the projection-valued spectral measures associated with A_j.

For $z \in \mathbb{C} \setminus \mathrm{Sp}(A)$ and $\phi = (\phi_j) \in \mathcal{H}$, we can write (using the functional calculus):

$$\langle (A - zI)^{-1}\phi, \phi\rangle = \sum_{j=1}^\infty \langle (A_j - zI)^{-1}\phi_j, \phi_j\rangle = \sum_{j=1}^\infty \int_{\mathbb{R}} \frac{1}{\lambda - z}\,\mathrm{d}\mu_{\phi_j}^{(j)}(\lambda)$$

where $\mu_{\phi_j}^{(j)}(U) = \langle \mathcal{E}_{A_j}(U)\phi_j, \phi_j\rangle$. Then we have

$$\int_{\mathbb{R}} \frac{1}{\lambda - z}\,\mathrm{d}\mu_\phi(\lambda) = \int_{\mathbb{R}} \frac{1}{\lambda - z}\,\mathrm{d}\left(\sum_{j=1}^\infty \mu_{\phi_j}^{(j)}\right)(\lambda).$$

Since a finite positive Borel measure on $\mathbb{R}$ is determined by its Borel transform, we have $\mu_\phi = \sum_{j=1}^\infty \mu_{\phi_j}^{(j)}$.

If μ_ϕ is absolutely continuous, then for each Lebesgue null $A \subset \mathbb{R}$, we have $\sum_{j=1}^\infty \mu_{\phi_j}^{(j)}(A) = 0$. Therefore, $\mu_{\phi_j}^{(j)}(A) = 0$ for each j and so each $\mu_{\phi_j}^{(j)}$ is absolutely continuous. Conversely, if each $\mu_{\phi_j}^{(j)}$ is absolutely continuous, then $\sum_{j=1}^\infty \mu_{\phi_j}^{(j)}(A) = 0$ for each Lebesgue null $A \subset \mathbb{R}$. Hence, $\phi \in \mathcal{H}_{\mathrm{ac}}$ if and only if $\phi_j \in \mathcal{H}_{j,\mathrm{ac}}$ for each j and so $\mathcal{H}_{\mathrm{ac}} = \oplus_{j=1}^\infty \mathcal{H}_{j,\mathrm{ac}}$.

Now suppose that μ_ϕ is singular continuous, then there exists a Lebesgue null set B such that $\mu_\phi(\mathbb{R}\setminus B) = 0$ and we have $\mu_\phi(\{x\}) = 0$ for each $x \in \mathbb{R}$. Then as above, we have $\mu_{\phi_j}^{(j)}(\mathbb{R}\setminus B) = 0$ and $\mu_{\phi_j}^{(j)}(\{x\}) = 0$, and so each $\mu_{\phi_j}^{(j)}$ is singular continuous. Conversely, if $\mu_{\phi_j}^{(j)}$ is concentrated on the Lebesgue null B_j for each j, then $B = \cup_j B_j$ is Lebesgue null and μ_ϕ is concentrated there. So μ_ϕ is singular continuous. So $\phi \in \mathcal{H}_{\mathrm{sc}}$ if and only if $\phi_j \in \mathcal{H}_{j,\mathrm{sc}}$ for each j. Hence, $\mathcal{H}_{\mathrm{sc}} = \oplus_{j=1}^\infty \mathcal{H}_{j,\mathrm{sc}}$.

By taking orthogonal complements, we also have $\mathcal{H}_{\mathrm{pp}} = \oplus_{j=1}^\infty \mathcal{H}_{j,\mathrm{pp}}$. It now follows that $\mathcal{H}_{\mathrm{s}} = \oplus_{j=1}^\infty \mathcal{H}_{j,\mathrm{s}}$ and $\mathcal{H}_{\mathrm{c}} = \oplus_{j=1}^\infty \mathcal{H}_{j,\mathrm{c}}$.

It follows that

$$\mathrm{Sp}_\diamond(A) = \mathrm{Sp}(\mathcal{P}_\diamond A \mathcal{P}_\diamond{}^*) = \mathrm{Sp}\left(\bigoplus_{j=1}^\infty \mathcal{P}_\diamond^{(j)} A_j \mathcal{P}_\diamond^{(j)*}\right) = \mathrm{Cl}\left(\cup_j \mathrm{Sp}_\diamond(\mathcal{P}_\diamond^{(j)} A_j \mathcal{P}_\diamond^{(j)*})\right) = \mathrm{Cl}(\cup_j \mathrm{Sp}_\diamond(A_j)).$$

Exercise 6.2

Take a discrete Schrödinger operator A_1 on $\ell^2(\mathbb{Z})$ with purely singular continuous spectrum equal to $[-3,3]$. Then $A_1/3$ has purely singular continuous spectrum on $[-1,1]$. Let H_0 be the free discrete Schrödinger operator with zero potential, then $A_2 = H_0/2$ has purely absolutely continuous spectrum equal to $[-1,1]$. Let $\{x_n\}_{n\in\mathbb{Z}}$ be a dense sequence in $[-1,1]$. Define the diagonal operator $[A_3]_{jj} = x_j$ on $\ell^2(\mathbb{Z})$. Then A_3 has pure point spectrum equal to $[-1,1]$. Then $A = A_1 \oplus A_2 \oplus A_3$, by the previous exercise, is such that $\mathrm{Sp}(A) = \mathrm{Sp}_{\mathrm{ac}}(A) = \mathrm{Sp}_{\mathrm{sc}}(A) = \mathrm{Sp}_{\mathrm{pp}}(A) = [-1,1]$. We can represent A as a banded operator on $\ell^2(\mathbb{Z})$ by interlacing. That is, let $\{e_j^{(k)}\}_{j\in\mathbb{Z}}$ be copies of the canonical basis vectors for $k = 1,2,3$, where A_k acts on $\mathcal{H}_k$ with basis $\{e_j^{(k)}\}_{j\in\mathbb{Z}}$. We then consider the ordering $\ldots, e_{-1}^1, e_{-1}^2, e_{-1}^3, e_0^1, e_0^2, e_0^3, e_1^1, e_1^2, e_1^3, \ldots$.

Exercise 6.3

By the theorem stated in the exercise, for every $U \in \mathcal{U}_{\mathrm{per}}$,

$$\xi_v^{(\mathrm{c})}(U) = \|\mathcal{P}_{\mathrm{c}}\mathcal{E}(U)v\|^2 = \lim_{n\to\infty}\lim_{K\to\infty} \frac{1}{2K+1}\sum_{k=-K}^{K} \|(I - \mathcal{P}_n^*\mathcal{P}_n)A^k \chi_U(A)v\|^2. \tag{19}$$

Using property $\mathcal{R}_U$, and the results on computing the functional calculus for unitary operators, there exist arithmetic algorithms $\tilde{\Gamma}_{n_2,n_1}$ using Λ such that

$$\left|\|\tilde{\Gamma}_{n_2,n_1}(A,v,U,k)\|^2 - \|(I-\mathcal{P}_{n_2}^*\mathcal{P}_{n_2})A^k\chi_U(A)v\|^2\right| \leq \frac{C(A,v,U)}{n_1} \quad \forall (A,v,U)\in\Omega\times\mathcal{U}_{\mathrm{per}}, k\in\mathbb{N},$$

where $C(A,v,U)$ is a constant that may depend on A, v and U but not on n_1, n_2 or k. Then define

$$\Gamma_{n_2,n_1}(A,v,U) = \frac{1}{2n_1+1}\sum_{k=-n_1}^{n_1}\|\tilde{\Gamma}_{n_2,n_1}(A,v,U,k)\|^2.$$

Then

$$\left|\Gamma_{n_2,n_1}(A,v,U) - \frac{1}{2n_1+1}\sum_{k=-n_1}^{n_1}\|(I-\mathcal{P}_{n_2}^*\mathcal{P}_{n_2})A^k\chi_U(A)v\|^2\right| \leq \frac{1}{2n_1+1}\sum_{k=-n_1}^{n_1}\frac{C(A,v,U)}{n_1}$$

which tends to 0 as $n_1\to\infty$. The result then follows by taking the limit $n_2\to\infty$ and using (19); the Π_2^A rather than Δ_2^A classification follows as the $\mathcal{P}_{n_2}$ are increasing.

Exercise 6.4

Recall that

$$\xi_v^{\epsilon}(\theta_0) = -\frac{1}{2\pi}\mathrm{Re}\left(\left\langle (A-(1+\epsilon)e^{\mathrm{i}\theta_0}I)^{-1}v, (A^*+(1+\epsilon)e^{-\mathrm{i}\theta_0}I)v\right\rangle\right),$$

which can be computed using the resolvent. Also, for almost every $\varphi\in[-\pi,\pi]_{\mathrm{per}}$, $\lim_{\epsilon\downarrow 0}\xi_v^{\epsilon}(\varphi) = \rho_v(\varphi)$, for ρ_v the Radon–Nikodym derivative of $\xi_v^{(\mathrm{ac})}$. We want to compute

$$\xi_v^{(\mathrm{ac})}(U) = \int_U \rho_v(\varphi)\,\mathrm{d}\varphi.$$

Let $(A,v,U)\in\Omega\times\mathcal{U}_{\mathrm{per}}$. Then we have access to an at most countable disjoint union $U=\cup_m(a_m(U),b_m(U))$ with $a_m(U),b_m(U)\in[-\pi,\pi]_{\mathrm{per}}$; from this, we can construct a sequence $\{g_{n_1}\}_{n_1=1}^{\infty}$ of non-negative, continuous, piecewise affine functions on $[-\pi,\pi]_{\mathrm{per}}$, bounded by 1 such that $g_{n_1}(\varphi)\uparrow\chi_U(\varphi)$ as $n_1\to\infty$ for all $\varphi\in[-\pi,\pi]_{\mathrm{per}}$. We can also construct a sequence $\{f_{n_2}\}_{n_2=1}^{\infty}$ of non-negative, continuous, piecewise affine functions on $\mathbb{R}$, bounded by 1 and of compact support, such that $f_{n_2}(x)\uparrow 1$ as $n_2\to\infty$ for all $x\in\mathbb{R}$. Let

$$I(n_2,n_1) = \int_{-\pi}^{\pi} g_{n_1}(\varphi)\xi_v^{1/n_1}(\varphi)f_{n_2}(\xi_v^{1/n_1}(\varphi))\,\mathrm{d}\varphi.$$

As $[-\pi,\pi]_{\mathrm{per}}$ is compact, $g_{n_1}(\varphi)$ is bounded by 1 for all φ and f_{n_2} has compact support, the integrand is uniformly bounded in n_1 for each fixed n_2. Then as $\xi_v^{1/n_1}(\varphi)$ converges pointwise almost everywhere to $\rho_v(\varphi)$, $g_{n_1}(\varphi)$ converges pointwise to $\chi_U(\varphi)$ and f_{n_2} is continuous, by the dominated convergence theorem,

$$\lim_{n_1\to\infty} I(n_2,n_1) = \int_U \rho_v(\varphi)f_{n_2}(\rho_v(\varphi))\,\mathrm{d}\varphi.$$

Then as the f_{n_2} are increasing and bounded by 1, using the dominated convergence theorem again we get that

$$\lim_{n_2\to\infty}\lim_{n_1\to\infty} I(n_2,n_1) = \int_U \rho_v(\varphi)\,\mathrm{d}\varphi = \xi_v^{(\mathrm{ac})}(U)$$

with the convergence monotonic from below. To finish the proof, note that $I(n_2,n_1)$ can be approximated to accuracy $O(1/n_1)$ using Λ together with the arguments from the unitary analogue of Theorem 4.3.11.

Exercise 6.5

Throughout, we consider the self-adjoint case and assume the requirements of Theorem 6.2.1. The unitary case is dealt with analogously.

Lower bounds: We begin by adapting the proof of Theorem 6.2.2 for the pure point spectrum. It suffices to consider a fixed vector e_1. Suppose for a contradiction that there exists a sequence of general algorithms, $\{\Gamma_n\}$, using Λ such that, for all $A \in \Omega_{\mathrm{DS}}$, $\Gamma_n(A)$ converges weakly to $\mu^{(\mathrm{pp})}_{e_1,A}$, i.e., for all bounded, continuous functions $\phi : \mathbb{R} \to \mathbb{C}$,

$$\lim_{n\to\infty} \int_{\mathbb{R}} \phi(y)\, \mathrm{d}(\Gamma_n(A))(y) = \int_{\mathbb{R}} \phi(y)\, \mathrm{d}\mu^{(\mathrm{pp})}_{e_1,A}(y).$$

In particular, if we take $\phi = 1$ the constant function, then the above says that

$$\lim_{n\to\infty} \Gamma_n(A)(\mathbb{R}) = \mu^{(\mathrm{pp})}_{e_1}(\mathbb{R}).$$

From here the proof follows exactly as in the book, replacing $(-4, 4)$ with $\mathbb{R}$.

Next we consider the absolutely continuous spectrum using Theorem 6.2.4; the singular continuous spectrum follows an almost identical argument. As before, we assume that there exists a sequence of general algorithms, $\{\Gamma_n\}$, using Λ, such that for all $A \in \Omega_{\mathrm{DS}}$ and for all bounded, continuous functions $\phi : \mathbb{R} \to \mathbb{C}$,

$$\lim_{n\to\infty} \int_{\mathbb{R}} \phi(y)\, \mathrm{d}(\Gamma_n(A))(y) = \int_{\mathbb{R}} \phi(y)\, \mathrm{d}\left[\mu^{(\mathrm{ac})}_{e_0,A} + \mu^{(\mathrm{ac})}_{e_1,A}\right](y).$$

Then choosing $\phi = 1$ the constant function, we have that

$$\lim_{n\to\infty} \Gamma_n(A)(\mathbb{R}) = \mu^{(\mathrm{ac})}_{e_0,A}(\mathbb{R}) + \mu^{(\mathrm{ac})}_{e_1,A}(\mathbb{R}).$$

From here, the rest of the proof is identical, replacing $(-2, 2)$ with $\mathbb{R}$.

Upper bounds: Let $\diamond = \mathrm{ac}$ or pp, $a \in \{-\infty\} \cup \mathbb{R}$, and $b \in \mathbb{R}$ with $b > a$. If $\diamond = \mathrm{ac}$, then $\mu^{(\diamond)}_v((a, b)) = \mu^{(\diamond)}_v((a, b])$ as the Lebesgue measure of singletons is zero. If $\diamond = \mathrm{pp}$, then there exists a height-one arithmetic tower for computing $\mu^{(\diamond)}_v(\{b\})$, which we can subtract from the Σ^A_2-tower for $\mu^{(\diamond)}_v((a, b))$. It follows that, in either case, there is a Σ^A_2-tower for computing $\mu^{(\diamond)}_v((a, b])$. We let $\hat{\mu}_{n_2,n_1}$ be the two limit algorithm that realises the Σ^A_2 convergence:

$$\lim_{n_2\to\infty} \lim_{n_1\to\infty} \hat{\mu}_{n_2,n_1}((a, b]) = \mu^{(\diamond)}_v((a, b]). \tag{20}$$

We now consider a sequence of partitions of the real line as follows. Let $\{x_j\}_{j\in\mathbb{N}}$ be a dense sequence in $\mathbb{R}$. Given an integer $n \in \mathbb{N}$, we let $\{d_{n,1}, \dots, d_{n,n+1}\}$ be a rearrangement of $\{x_j\}_{j=1}^{n+1}$ such that $d_{n,1} < d_{n,2} < \cdots < d_{n,n+1}$. We then define the partition P_n as

$$P_n = \{(-\infty, d_{n,1}], (d_{n,1}, d_{n,2}], \dots, (d_{n,n+1}, \infty)\}.$$

For $n_1, n_2, k \in \mathbb{N}$, we define the measure

$$\mu'_{k,n_2,n_1} = \hat{\mu}_{n_2,n_1}((-\infty, d_{k,1}])\delta_{d_{k,1}} + \sum_{j=1}^{k} \hat{\mu}_{n_2,n_1}((d_{k,j}, d_{k,j+1}])\delta_{d_{k,j+1}}$$

with corresponding cumulative distribution function

$$F'_{k,n_2,n_1}(x) = \mu'_{k,n_2,n_1}((-\infty, x]).$$

We now define the cumulative distribution function

$$F_{n_2,n_1}(x) = \max_{1\le k\le n_2} F'_{k,n_2,n_1}(x).$$

This cumulative distribution function induces a discrete measure, μ_{n_2,n_1}, whose support is contained in $\{x_j\}_{j=1}^{n_2+1}$. As $n_1 \to \infty$, μ'_{k,n_2,n_1} converges weakly (and setwise) to

$$\mu'_{k,n_2} = \hat{\mu}_{n_2}((-\infty, d_{k,1}])\delta_{d_{k,1}} + \sum_{j=1}^{k} \hat{\mu}_{n_2}((d_{k,j}, d_{k,j+1}])\delta_{d_{k,j+1}}.$$

Moreover, the function F'_{k,n_2,n_1} converges uniformly to F'_{k,n_2} with $F'_{k,n_2}(x) = \mu'_{k,n_2}((-\infty, x])$. Hence, F_{n_2,n_1} converges uniformly to the function

$$F_{n_2}(x) = \max_{1\leq k\leq n_2} F'_{k,n_2}(x) \leq \mu_v^{(\diamond)}((-\infty, x]),$$

where the final inequality holds due to the Σ_2^A convergence in (20). Let $F(x) = \mu_v^{(\diamond)}((-\infty, x])$ be the cumulative distribution function of $\mu_v^{(\diamond)}$ and let x be a continuity point of F. Given $\epsilon > 0$, there exists $y < x$ with $F(y) \geq F(x) - \epsilon$. There exists some $x_N \in (y, x)$. If $n_2 \geq N$, then

$$F_{n_2}(x) \geq F'_{N,n_2}(x_N) = \hat{\mu}_{n_2}((-\infty, d_{N,1}]) + \sum_{j:d_{N,j+1}\leq x_N} \hat{\mu}_{n_2}((d_{N,j}, d_{N,j+1}]).$$

As $n_2 \to \infty$, the quantity on the right-hand side converges to $\mu_v^{(\diamond)}((-\infty, x_N]) = F(x_N)$. It follows that

$$\liminf_{n_2\to\infty} F_{n_2}(x) \geq F(x_N) \geq F(y) \geq F(x) - \epsilon.$$

Since $\epsilon > 0$ was arbitrary, we see that F_{n_2} converges pointwise to F at every continuity point of F. It follows that $\lim_{n_1\to\infty} \mu_{n_2,n_1}$ converges weakly to $\mu_v^{(\diamond)}$ as $n_2 \to \infty$.

We now adapt this argument to $\diamond \in \{\mathrm{sc}, \mathrm{c}, \mathrm{s}\}$, where instead we have a Π_2^{A}-tower for computing $\mu_v^{(\diamond)}((a, b])$:

$$\lim_{n_2\to\infty} \lim_{n_1\to\infty} \hat{\mu}_{n_2,n_1}((a, b]) = \mu_v^{(\diamond)}((a, b]). \tag{21}$$

We now define the measures

$$\mu'_{k,n_2,n_1} = \hat{\mu}_{n_2,n_1}((d_{k,k+1}, +\infty))\delta_{d_{k,k+1}} + \sum_{j=1}^{k} \hat{\mu}_{n_2,n_1}((d_{k,j}, d_{k,j+1}])\delta_{d_{k,j}}$$

and the functions

$$G'_{k,n_2,n_1}(x) = \hat{\mu}_{n_2,n_1}((-\infty, d_{k,1}]) + \mu'_{k,n_2,n_1}((-\infty, x]),$$

which are not quite cumulative distribution functions. We also define the functions

$$G_{n_2,n_1}(x) = \min_{1\leq k\leq n_2} G'_{k,n_2,n_1}(x).$$

As $n_1 \to \infty$, these functions converge uniformly to

$$G_{n_2}(x) = \min_{1\leq k\leq n_2} \hat{\mu}_{n_2}((-\infty, d_{k,1}]) + \mu'_{k,n_2}((-\infty, x]) \geq \mu_v^{(\diamond)}((-\infty, x]),$$

where the final inequality holds due to the Π_2^A convergence in (21). Fix $x \in \mathbb{R}$ and let $\epsilon > 0$. Since $\mu_v^{(\diamond)}((-\infty, x])$ is right-continuous, there exists $y > x$ with $\mu_v^{(\diamond)}((-\infty, y]) \leq \mu_v^{(\diamond)}((-\infty, x]) + \epsilon$. There exists some $N \in \mathbb{N}$ with $d_{N,l} \in (x, y)$ and $d_{N,l+1} < y$. If $n_2 \geq N$, then

$$G_{n_2}(x) \leq G_{n_2}(d_{N,l}) \leq \hat{\mu}_{n_2}((-\infty, d_{N,1}]) + \sum_{j=1}^{l} \hat{\mu}_{n_2}((d_{N,j}, d_{N,j+1}]).$$

As $n_2 \to \infty$, the quantity on the right-hand side converges to $\mu_v^{(\diamond)}((-\infty, d_{N,l+1}]) \leq G(y) = \int_{(-\infty,y]} 1 \, \mathrm{d}\mu_v^{(\diamond)}(s)$. It follows that

$$\limsup_{n_2\to\infty} G_{n_2}(x) \leq G(y) \leq G(x) + \epsilon.$$

Since $\epsilon > 0$ and x were arbitrary, we see that G_{n_2} converges pointwise to G everywhere.

Exercise 6.6

Let $(A, \nu, U) \in \Omega \times \mathcal{U}_{\mathbb{R}}$ with

$$U = \bigcup_m (a_m, b_m),$$

where $a_m, b_m \in \mathbb{R} \cup \{\pm\infty\}$ and the disjoint union is at most countable. Without loss of generality, we assume that the union is over $m \in \mathbb{N}$. We have

$$\lim_{l\to\infty} \frac{\pi l}{2} \int_{\mathbb{R}} \phi(y) \chi_{\{x:|H^{\mathbb{R}}_{\mu_\nu}(x)|\geq l\}}(y)\, \mathrm{d}y = \int_{\mathbb{R}} \phi(y)\, \mathrm{d}\mu_\nu^{(\mathrm{s})}(y), \tag{22}$$

for every bounded continuous function $\phi : \mathbb{R} \to \mathbb{C}$. Due to the possibility of point spectra at the endpoints a_m, b_m, we cannot simply replace ϕ by χ_U in the above limit. However, this can be overcome in the following manner.

Let ∂U denote the boundary of U defined by $\overline{U} \setminus U$ and let ν denote the measure $\mu_\nu\restriction_{\partial U}$. Let $\{f_l\}_{l\in\mathbb{N}}$ denote a pointwise increasing sequence of continuous functions, converging everywhere up to χ_U, such that the support of each f_l is contained in

$$[-l, l] \bigcap \left(\bigcup_{m=1}^{l} \left(a_m + 1/\sqrt{l}, b_m - 1/\sqrt{l}\right) \right).$$

Such a sequence exists (and can easily be explicitly constructed) precisely because U is open. We first claim that

$$\lim_{l\to\infty} \frac{\pi l}{2} \int_{\mathbb{R}} f_l(t) \chi_{\{x:|H^{\mathbb{R}}_{\mu_\nu}(x)|\geq l\}}(t)\, \mathrm{d}t = \mu_\nu^{(\mathrm{s})}(U). \tag{23}$$

To see this, note that for each $k \in \mathbb{N}$, the following inequalities hold

$$\liminf_{l\to\infty} \frac{\pi l}{2} \int_{\mathbb{R}} f_l(t) \chi_{\{x:|H^{\mathbb{R}}_{\mu_\nu}(x)|\geq l\}}(t)\, \mathrm{d}t \geq \liminf_{l\to\infty} \frac{\pi l}{2} \int_{\mathbb{R}} f_k(t) \chi_{\{x:|H^{\mathbb{R}}_{\mu_\nu}(x)|\geq l\}}(t)\, \mathrm{d}t = \int_{\mathbb{R}} f_k(t)\, \mathrm{d}\mu_\nu^{(\mathrm{s})}(t),$$

where the last equality is due to (22). Taking $k \to \infty$ and using the dominated convergence theorem yields

$$\liminf_{l\to\infty} \frac{\pi l}{2} \int_{\mathbb{R}} f_l(t) \chi_{\{x:|H^{\mathbb{R}}_{\mu_\nu}(x)|\geq l\}}(t)\, \mathrm{d}t \geq \mu_\nu^{(\mathrm{s})}(U),$$

so we are left with proving a similar bound for the limit supremum. Note that every point in the support of f_l is of distance at least $1/\sqrt{l}$ from ∂U. It follows that there exists a constant C independent of l such that for all $t \in \mathrm{supp}(f_l)$,

$$\left|H^{\mathbb{R}}_\nu(t)\right| \leq C\sqrt{l}$$

Now let $\epsilon \in (0, 1)$. Then, for large l, $l - C\sqrt{l} \geq (1 - \epsilon)l$ and hence

$$\mathrm{supp}(f_l) \cap \{w : |H^{\mathbb{R}}_{\mu_\nu}(w)| \geq l\} \subset \mathrm{supp}(f_l) \cap \{w : |H^{\mathbb{R}}_{\mu_\nu - \nu}(w)| \geq (1 - \epsilon)l\}. \tag{24}$$

Now let f be a bounded continuous function such that $f \geq \chi_U$. Then using (24),

$$\begin{aligned}
\limsup_{l\to\infty} \frac{\pi l}{2} \int_{\mathbb{R}} f_l(t) \chi_{\{x:|H^{\mathbb{R}}_{\mu_\nu}(x)|\geq l\}}(t)\, \mathrm{d}t &\leq \limsup_{l\to\infty} \frac{1}{1-\epsilon} \frac{\pi(1-\epsilon)l}{2} \int_{\mathbb{R}} f_l(t) \chi_{\{x:|H^{\mathbb{R}}_{\mu_\nu-\nu}(x)|\geq(1-\epsilon)l\}}(t)\, \mathrm{d}t \\
&\leq \limsup_{l\to\infty} \frac{1}{1-\epsilon} \frac{\pi(1-\epsilon)l}{2} \int_{\mathbb{R}} f(t) \chi_{\{x:|H^{\mathbb{R}}_{\mu_\nu-\nu}(x)|\geq(1-\epsilon)l\}}(t)\, \mathrm{d}t \\
&= \frac{1}{1-\epsilon} \int_{\mathbb{R}} f(t)\, \mathrm{d}\left[\mu_\nu^{(\mathrm{s})} - \nu^{(\mathrm{s})}\right](t).
\end{aligned}$$

Now we let $f \downarrow \chi_{\overline{U}}$, with pointwise convergence everywhere. This is possible since the complement of $\overline{U}$ is open. By the dominated convergence theorem, and since ϵ was arbitrary, this yields

$$\limsup_{l\to\infty} \frac{\pi l}{2} \int_{\mathbb{R}} f_l(t) \chi_{\{x:|H^{\mathbb{R}}_{\mu_\nu}(x)|\geq l\}}(t)\, \mathrm{d}t \leq [\mu_\nu^{(\mathrm{s})} - \nu^{(\mathrm{s})}](\overline{U}) = \mu_\nu^{(\mathrm{s})}(U),$$

where the last equality follows from the definition of ν. The convergence in (23) now follows.

Let χ_n be a sequence of non-negative continuous piecewise affine functions on $\mathbb{R}$, bounded by 1 and such that $\chi_n(t) = 0$ if $t \leq n-1$ and $\chi_n(t) = 1$ if $t \geq n+1$. Consider the integrals

$$I(n,m) = \frac{\pi n}{2}\int_{\mathbb{R}} f_n(t)\chi_n(|F_m(t)|)\,\mathrm{d}t,$$

where $F_m(t)$ is an approximation of

$$\frac{-1}{\pi}\mathrm{Re}\left(\langle (A-(t+\mathrm{i}/m)I)^{-1}v, v\rangle\right)$$

to pointwise accuracy $O(m^{-1})$ over $t \in [-n,n]$. Note that a suitable piecewise affine function f_n can be constructed using Λ, as can suitable χ_n, and a suitable approximation function F_m can be pointwise evaluated using Λ. To define f_n, we may use suitable piecewise affine functions on each interval $[-n,n]\cap(a_m+1/\sqrt{n}, b_m-1/\sqrt{n})$. It follows that there exists arithmetic algorithms $\Gamma_{n,m}(A,v,U)$ using Λ such that

$$\left|I(n,m) - \Gamma_{n,m}(A,v,U)\right| \leq \frac{C(A,v,U)}{m}.$$

The dominated convergence theorem implies that

$$\lim_{m\to\infty}\Gamma_{n,m}(A,v,U) = \lim_{m\to\infty} I(n,m) = \frac{\pi n}{2}\int_{\mathbb{R}} f_n(t)\chi_n(|H^{\mathbb{R}}_{\mu_v}(t)|)\,\mathrm{d}t.$$

Note that continuity of the χ_n is needed to gain convergence almost everywhere and prevent possible oscillations about the level set $\{H^{\mathbb{R}}_{\mu_v}(t) = n\}$. We also have that

$$\chi_{\{w:|H^{\mathbb{R}}_{\mu_v}(w)|\geq n+1\}}(t) \leq \chi_n(|H^{\mathbb{R}}_{\mu_v}(t)|) \leq \chi_{\{w:|H^{\mathbb{R}}_{\mu_v}(w)|\geq n-1\}}(t).$$

The same arguments used to prove (23) therefore show that

$$\lim_{n\to\infty}\frac{\pi n}{2}\int_{\mathbb{R}} f_n(t)\chi_n(|H^{\mathbb{R}}_{\mu_v}(t)|)\,\mathrm{d}t = \mu_v^{(\mathrm{s})}(U).$$

Hence, $\lim_{n\to\infty}\lim_{m\to\infty}\Gamma_{n,m}(A,v,U) = \mu_v^{(\mathrm{s})}(U)$, which shows that $\{\Xi_{\mu,\mathrm{s}}, \Omega\times\mathcal{U}_{\mathbb{R}}, \mathbb{R}, \Lambda\} \in \Delta_3^A$.

Exercise 6.7

Let $(A,v,U) \in \Omega\times\mathcal{U}_{\mathrm{per}}$ with

$$U = \bigcup_m (a_m, b_m)$$

where $a_m, b_m \in [-\pi,\pi]_{\mathrm{per}}$ and the disjoint union is at most countable. Without loss of generality, assume that the union is over $m\in\mathbb{N}$. We have that

$$\lim_{l\to\infty}\frac{\pi l}{2}\int_{[-\pi,\pi]_{\mathrm{per}}}\phi(\theta)\chi_{\{t:|H^{\mathbb{T}}_{\xi_v}(t)|\geq l\}}(\theta)\,\mathrm{d}\theta = \int_{[-\pi,\pi]_{\mathrm{per}}}\phi(\theta)\,\mathrm{d}\xi_v^{(\mathrm{s})}(\theta)$$

for every periodic continuous function $\phi : [-\pi,\pi]_{\mathrm{per}} \to \mathbb{C}$. Let $\partial U = \overline{U}\setminus U$ and $\nu = \xi_v\restriction_{\partial U}$. Let $\{f_l\}_{l\in\mathbb{N}}$ denote a pointwise increasing sequence of continuous functions, converging everywhere up to χ_U, such that the support of each f_l is contained in

$$\bigcup_{m=1}^{l}\left(a_m+1/\sqrt{l}, b_m-1/\sqrt{l}\right).$$

By adapting the self-adjoint case in the previous exercise, we have that

$$\lim_{l\to\infty}\frac{\pi l}{2}\int_{[-\pi,\pi]_{\mathrm{per}}} f_l(\theta)\chi_{\{t:|H^{\mathbb{T}}_{\xi_v}(t)|\geq l\}}(\theta)\,\mathrm{d}\theta = \xi_v^{(\mathrm{s})}(U).$$

It remains to turn this result into an arithmetic algorithm. Let χ_n be a sequence of non-negative continuous piecewise affine functions on $\mathbb{R}$, bounded by 1 and such that $\chi_n(t) = 0$ if $t \leq n-1$ and $\chi_n(t) = 1$ if $t \geq n+1$. Consider the integrals

$$I(n,m) = \frac{\pi n}{2}\int_{[-\pi,\pi]_{\mathrm{per}}} f_n(\theta)\chi_n(|F_m(\theta)|)\,\mathrm{d}\theta,$$

where $F_m(\theta)$ is an approximation of

$$\frac{1}{2\pi}\mathrm{Im}\left(\langle (A+(1+1/m)e^{\mathrm{i}\theta}I)(A-(1+1/m)e^{\mathrm{i}\theta}I)^{-1}v,v\rangle\right)$$

to pointwise accuracy $O(m^{-1})$. Note that a suitable piecewise affine function f_n can be constructed using Λ, as can suitable χ_n, and a suitable approximation function F_m can be pointwise evaluated using Λ. To define f_n, we can define the function by suitable piecewise affine functions on each interval $(a_m + 1/\sqrt{n}, b_m - 1/\sqrt{n})$. It follows that there exists arithmetic algorithms $\Gamma_{n,m}(A,v,U)$ using Λ such that

$$|I(n,m) - \Gamma_{n,m}(A,v,U)| \leq \frac{C(A,v,U)}{m}.$$

Then by the dominated convergence theorem,

$$\lim_{m\to\infty}\Gamma_{n,m}(A,v,U) = \lim_{m\to\infty} I(n,m) = \frac{\pi n}{2}\int_{[-\pi,\pi]_{\mathrm{per}}} f_n(\theta)\chi_n(|H^{\mathbb{T}}_{\xi_v}(\theta)|)\mathrm{d}\theta.$$

We also have that

$$\chi_{\{t:|H^{\mathbb{T}}_{\xi_v}(t)|\geq n+1\}}(\theta) \leq \chi_n(|H^{\mathbb{T}}_{\xi_v}(\theta)|) \leq \chi_{\{t:|H^{\mathbb{T}}_{\xi_v}(t)|\geq n-1\}}(\theta)$$

and so the same arguments as used previously show that

$$\lim_{n\to\infty}\frac{\pi n}{2}\int_{[-\pi,\pi]_{\mathrm{per}}} f_n(\theta)\chi_n(|H^{\mathbb{T}}_{\xi_v}(\theta)|)\mathrm{d}\theta = \xi^{(\mathrm{s})}_v(U).$$

Hence $\lim_{n\to\infty}\lim_{m\to\infty}\Gamma_{n,m}(A,v,U) = \xi^{(\mathrm{s})}_v(U)$, and so $\{\Xi_{\xi,\mathrm{s}}, \Omega\times\mathcal{U}_{\mathrm{per}}, \mathbb{R}, \Lambda\} \in \Delta^A_3$.

Exercise 6.8

It suffices to consider the unit vector e_1 and $U = [-\pi,\pi]_{\mathrm{per}}$. Suppose, for a contradiction, that there exists a sequence of general algorithms, $\{\Gamma_n\}$, using Λ such that for all $A \in \Omega_{\mathrm{CMV}}$,

$$\lim_{n\to\infty}\Gamma_n(A) = \xi^{(\mathrm{pp})}_{e_1,A}([-\pi,\pi]_{\mathrm{per}}).$$

Let A_1 be a CMV matrix with Verblunsky coefficients $\kappa^{(1)}_{m!} = 1/2$ for $m \geq 1$ and $\kappa^{(1)}_n = 0$ otherwise. Then ξ_{e_1,A_1} is purely singular continuous (see Theorems quoted in exercise) and so $\xi^{(\mathrm{pp})}_{e_1,A_1}([-\pi,\pi]_{\mathrm{per}}) = 0$. Hence, there exists n_1 such that $\Gamma_{n_1}(A_1) < 1/4$; moreover, there exists N_1 such that $\Gamma_{n_1}(A_1)$ only depends on the Verblunsky coefficients $\{\kappa^{(1)}_j\}_{j=1}^{N_1}$ (using the structure of CMV matrices). Now let A_2 be a CMV matrix whose Verblunsky coefficients are a realisation of a Bernoulli random process with equally likely outcomes 0 or 1/2, $\xi^{(\mathrm{pp})}_{e_1,A_2}([-\pi,\pi]_{\mathrm{per}}) = 1$, and $\kappa^{(2)}_j = \kappa^{(1)}_j$ for $1 \leq j \leq N_1$; such a CMV matrix exists as there is nonzero chance for the Verblunsky coefficients to agree at finitely many points. Then by consistency of general algorithms, $\Gamma_{n_1}(A_1) = \Gamma_{n_1}(A_2)$. However, since $\xi^{(\mathrm{pp})}_{e_1,A_2}([-\pi,\pi]_{\mathrm{per}}) = 1$, there exists $n_2 > n_1$ such that $\Gamma_{n_2}(A_2) > 3/4$, and $N_2 > N_1$ such that $\Gamma_{n_2}(A_2)$ only depends on the Verblunsky coefficients $\{\kappa^{(2)}_j\}_{j=1}^{N_2}$. We repeat this process by induction; for example, we let A_3 be a CMV matrix with $\kappa^{(3)}_j = \kappa^{(2)}_j$ for $1 \leq j \leq N_2$, $\kappa^{(3)}_{m!} = 1/2$ for $m \geq m_3$ such that $m_3! \geq N_2$, and $\kappa^{(3)}_n = 0$ otherwise. Then if we define Verblunsky coefficients $\kappa_j = \kappa^{(k)}_j$ for $1 \leq j \leq N_k$ and corresponding CMV matrix A, we see that $\Gamma_n(A)$ cannot converge, a contradiction.

Exercise 6.9

We deal with $\diamond = \text{ac}$, and the result for $\diamond = \text{sc}$ follows an almost identical argument. To prove that $\{\Xi_{\xi,\text{ac}}, \Omega_{\text{CMV}} \times \{e_n : n \in \mathbb{N}\} \times \mathcal{U}_{\text{per}}, \mathbb{R}, \Lambda\} \notin \Delta_2^G$, we reduce a decision problem with SCI > 1 to the computation of $\Xi_{\xi,\text{ac}}$. Let $\mathcal{M}_{\text{dec}}$ be the space $\{0,1\}$ equipped with the discrete topology, let Ω' denote the collection of all infinite sequences $\{a_j\}_{j\in\mathbb{N}}$ with entries $a_j \in \{0,1\}$ and consider the problem function $\Xi' : \Omega' \to \mathcal{M}_{\text{dec}}$ with

$$\Xi'(\{a_j\}) = \begin{cases} 1, & \text{if } \{a_j\} \text{ has infinitely many nonzero entries,} \\ 0, & \text{otherwise.} \end{cases}$$

Equip Ω' with the set Λ' of evaluation functions that evaluate $\{a_j\}$ component-wise. It is easy to see that $\{\Xi', \Omega', \mathcal{M}_{\text{dec}}, \Lambda'\} \notin \Delta_2^G$. Suppose for a contradiction that $\{\Gamma_n\}$ is a Δ_2^G-tower of algorithms such that

$$\lim_{n\to\infty} \Gamma_n(A) = \xi^{(\text{ac})}_{e_1,A}([-\pi,\pi]_{\text{per}}) \quad \forall A \in \Omega_{\text{CMV}}.$$

For convenience, we have added a subscript A to the notation of the spectral measure. We will gain a contradiction by using the supposed tower to solve $\{\Xi', \Omega', \mathcal{M}_{\text{dec}}, \Lambda'\}$.

Given $\{a_j\}$, define a CMV matrix $A(\{a_j\})$ with Verblunsky coefficients given by $\kappa_{m!} = a_m/2$ for $m \geq 1$, and $\kappa_j = 0$ otherwise. If $\Xi'(\{a_j\}) = 0$, then $\sum_{j=0}^{\infty} |\kappa_j|^2 = \frac{1}{4}\sum_{j=1}^{\infty} |a_j|^2 < \infty$ and so by the Theorem stated in the exercise, the spectrum of $A(\{a_j\})$ is purely absolutely continuous and so $\xi^{(\text{ac})}_{e_1,A(\{a_j\})}([-\pi,\pi]_{\text{per}}) = \xi_{e_1,A(\{a_j\})}([-\pi,\pi]_{\text{per}}) = 1$. Conversely, if $\Xi'(\{a_j\}) = 1$ then by the same theorem, the spectrum of $A(\{a_j\})$ is purely singular continuous and so $\xi^{(\text{ac})}_{e_1,A(\{a_j\})}([-\pi,\pi]_{\text{per}}) = 0$.

Now define

$$\hat{\Gamma}_n(\{a_j\}) = \begin{cases} 1, & \text{if } \Gamma_n(A(\{a_j\})) < 1/2, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\lim_{n\to\infty} \hat{\Gamma}_n(\{a_j\}) = \Xi'(\{a_j\})$. Moreover, the evaluation functions for computing matrix entries of $A(\{a_j\})$ can be computed using Λ'. Hence $\{\hat{\Gamma}_n\}$ can be converted into a Δ_2^G-tower of algorithms for solving $\{\Xi', \Omega', \mathcal{M}_{\text{dec}}, \Lambda'\}$, a contradiction.

Exercise 6.10

This is a straightforward adaptation, where we choose the level 0 intervals in the tree to cover $[-\pi,\pi]_{\text{per}}$.

Exercise 6.11

To prove the upper bound, let K be the Poisson kernel for the upper half plane. There is an arithmetic algorithm Γ that uses Λ such that

$$\left|\Gamma(A,j,\epsilon,n) - \frac{\epsilon}{K(0)}\int_{\mathbb{R}} K_\epsilon(1-x)\,\mathrm{d}\mu_{e_j,A}(x)\right| \leq \frac{1}{n} \quad \forall (A,j,\epsilon,n) \in \Omega_{\text{DS}} \times \mathbb{Z} \times \mathbb{R}_{>0} \times \mathbb{N}.$$

Note that, since K is positive,

$$\frac{\epsilon}{K(0)}\int_{\mathbb{R}} K_\epsilon(1-x)\,\mathrm{d}\mu_{e_j,A}(x) \geq \frac{\epsilon}{K(0)}\mu_{e_j,A}(\{1\})K_\epsilon(0) = \mu_{e_j,A}(\{1\}).$$

The results of Chapter 4 also imply that

$$\lim_{\epsilon\downarrow 0} \frac{\epsilon}{K(0)}\int_{\mathbb{R}} K_\epsilon(1-x)\,\mathrm{d}\mu_{e_j,A}(x) = \mu_{e_j,A}(\{1\}).$$

To achieve Π_1^A convergence, we set $\Gamma_n(A,j) = \Gamma(A,j,1/n,n) + 1/n$.

For the lower bound, suppose for a contradiction that $\{\widehat{\Xi}, \Omega_{\text{DS}} \times \mathbb{Z}, \mathbb{R}, \Lambda\} \in \Delta_1^G$ and that $\{\Gamma_n\}$ is a sequence of general algorithms solving the problem with error control. It follows that for each $j \in \mathbb{Z}$, there exists a sequence of general algorithms $\{\Gamma_n^j\}$ such that

$$\lim_{n\to\infty} \Gamma_n^j(A) = \begin{cases} 1, & \text{if } \widehat{\Xi}(A,j) > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Informally, these are described as follows. Fix j and consider the sharpest lower bound on $\widehat{\Xi}(A, j)$ computed by $\{\Gamma_m(A, j) : m \leq n\}$. If this is greater than 0 then set $\Gamma_n^j(A) = 1$, otherwise set $\Gamma_n^j(A) = 0$. It follows that $\Gamma_n^j(A)$ also converges from below. It holds that $1 \in \mathrm{Sp}_{\mathrm{p}}(A)$ if and only if $\widehat{\Xi}(A, j) > 0$ for some $j \in \mathbb{Z}$. Now define

$$\widehat{\Gamma}_n(A) = \sup_{j \leq n} \Gamma_n^j(A).$$

It is clear that this is a general algorithm using Λ. Furthermore,

$$\lim_{n \to \infty} \widehat{\Gamma}_n(A) = \begin{cases} 1, & \text{if } 1 \in \mathrm{Sp}_{\mathrm{p}}(A) \\ 0, & \text{otherwise,} \end{cases}$$

with convergence from below. Now we may choose a potential V such that (with $A = H_0 + V$) $1 \in \mathrm{Sp}_{\mathrm{p}}(A)$ (this can be achieved for example by taking a potential which induces pure point spectrum and shifting the operator accordingly). It follows that for large n, we have $\widehat{\Gamma}_n(A) = 1$. But the computation of $\widehat{\Gamma}_n(A)$ is only dependent on $V(j)$ for $|j| < N$ for some $N \in \mathbb{N}$. Define the potential by $V_0(j) = V(j)$ if $|j| < N$ and $V_0(j) = 0$ otherwise. It follows by consistency of general algorithms that $\widehat{\Gamma}_n(H_0 + V_0) = 1$. But since the potential has compact support, $1 \notin \mathrm{Sp}_{\mathrm{p}}(H_0 + V_0)$ and, hence, $\widehat{\Gamma}_n(H_0 + V_0) = 0$, the required contradiction.

Exercise 6.12

Code for the exercise can be found in "chapter6/UAM_spectral_type.m" in the repository. The unitary operator codes are simpler than their self-adjoint counterparts due to (a) not needing operator exponentials and (b) not needing quadrature approximations of integrals over time.

7 Chapter 7

Exercise 7.1

First, let $\Omega = \Omega_{\mathrm{D}}$, $\Omega_{\mathrm{B}} \cap \Omega_f \cap \Omega_{\mathrm{SA}}$ or $\Omega_{\mathrm{B}} \cap \Omega_f$ and p be a fixed polynomial with at least two distinct real roots. By Exercise 3.10, for each $A \in \Omega$, we can use Λ to evaluate the matrix entries of $p(A)$ to any desired accuracy in finitely many arithmetic operations and comparisons. Hence, we can approximate $\|\mathcal{P}_n p(A)\mathcal{P}_n^*\|$ to within accuracy $1/n$ using finitely many arithmetic operations and comparisons. Call this approximation $\tilde{\Gamma}_n(A)$, then $\Gamma_n(A) = \max\{\tilde{\Gamma}_n(A) - 1/n, 0\}$ is a Σ_1^A-tower for $\{\Xi_p, \Omega, \mathbb{R}_{\geq 0}, \Lambda\}$. It is also clear that the problem does not lie in Δ_1^G.

Now let $\Omega = \Omega_{\mathrm{B}} \cap \Omega_{\mathrm{SA}}$, and let $\Gamma_{n_2,n_1}(A)$ be an approximation of $\|\mathcal{P}_{n_2} p(\mathcal{P}_{n_1} A \mathcal{P}_{n_1}^*)\mathcal{P}_{n_2}^*\|$ to accuracy $1/n_1$, then

$$\lim_{n_1\to\infty} \Gamma_{n_2,n_1}(A) = \|\mathcal{P}_{n_2} p(A)\mathcal{P}_{n_2}^*\|.$$

Here, we have used the fact that convergence in the strong operator topology is preserved under composition of operators and, hence, under application of polynomials. It follows that $\{\Gamma_{n_2,n_1}\}$ is a Σ_2^A-tower for$\{\Xi_p, \Omega_{\mathrm{B}} \cap \Omega_{\mathrm{SA}}, \mathbb{R}_{\geq 0}, \Lambda\}$. Clearly, this extends to the class Ω_{B}.

For the lower bound for $\Omega' = \Omega_{\mathrm{B}} \cap \Omega_{\mathrm{SA}}$, we assume without loss of generality that the zeros of p are ± 1 and $|p(0)| > 1$ (the more general case is similar). Suppose for a contradiction that a height one tower, $\{\Gamma_n\}$, solves the problem. We will gain a contradiction by showing that $\Gamma_n(A)$ does not converge for an operator of the form,

$$A = \bigoplus_{r=1}^{\infty} B(z_1, \ldots, z_{l_r}),$$

for $l_r \geq r$, and define

$$C = \mathrm{diag}\{z_1, z_2, \ldots\} \in \Omega_{\mathrm{B}} \cap \Omega_{\mathrm{SA}}.$$

Since $\mathrm{Sp}(A) = \{-1, 1\}$, $\|p(A)\| = 0$. Now suppose that $l_1, \ldots, l_k$ have been chosen and consider the operator

$$B_k = B(z_1) \oplus \cdots \oplus B(z_1, \ldots, z_{l_k}) \oplus C.$$

The spectrum of B_k is $[-1, 1]$ so that $\|p(B_k)\| > 1$ and hence there exists $n(k) \geq k$ such that $\Gamma_{n(k)}(B_k) > 1/4$. But $\Gamma_{n(k)}(B_k)$ can only depend on the evaluations of the matrix entries $\{B_k\}_{ij} = \langle B_k e_j, e_i\rangle$ with $i, j \leq N(B_k, n(k))$ (as well as evaluations of the function f). If we choose $l_{k+1} > N(B_k, n(k))$ then $\Gamma_{n(k)}(A) = \Gamma_{n(k)}(B_k) > 1/4$. But $\Gamma_n(A)$ must converge to 0, a contradiction.

For the class $\Omega_{\mathrm{B}} \cap \Omega_g$, the lower bound we just proved clearly holds, and the upper bound also holds since $\Omega_{\mathrm{B}} \cap \Omega_g \subset \Omega_{\mathrm{B}}$. The lower bounds all hold if we allow the polynomial p to vary and treat it as an additional input. Similarly, the upper bounds can be adapted so that the towers of algorithms treat p as an additional input.

Exercise 7.2

Let $p_1, p_2, \ldots$ be an enumeration of the monic polynomials with rational coefficients and degree at least one, and set

$$a_n = \inf\left\{\|p_j\|_{L^\infty(\Gamma_n^{\mathrm{Sp}}(A))}^{\frac{1}{\deg(p_j)}} : j \in \{1, 2, \ldots, n\}\right\}.$$

Clearly,

$$a_n \geq \inf\left\{\|p_j\|_{L^\infty(\mathrm{Sp}(A))}^{\frac{1}{\deg(p_j)}} : j \in \{1, 2, \ldots, n\}\right\} \geq \inf_{\text{monic polynomial } p, \deg(p)\geq 1} \|p\|_{L^\infty(\mathrm{Sp}(A))}^{\frac{1}{\deg(p)}} = \Xi_{\mathrm{cap}}(A).$$

For each fixed N, we have

$$\lim_{n\to\infty} \inf\left\{\|p_j\|_{L^\infty(\Gamma_n^{\mathrm{Sp}}(A))}^{\frac{1}{\deg(p_j)}} : j \in \{1, 2, \ldots, N\}\right\} = \inf\left\{\|p_j\|_{L^\infty(\mathrm{Sp}(A))}^{\frac{1}{\deg(p_j)}} : j \in \{1, 2, \ldots, N\}\right\}.$$

Hence,

$$\limsup_{n\to\infty} a_n \leq \inf\left\{\|p_j\|_{L^\infty(\mathrm{Sp}(A))}^{\frac{1}{\deg(p_j)}} : j \in \mathbb{N}\right\} = \Xi_{\mathrm{cap}}(A),$$

where the final equality follows by density of the list $\{p_j\}$ in the set of monic polynomials. It follows that a_n converges to $\Xi_{\mathrm{cap}}(A)$ from above. To finish the proof of the Π_1^A classification, we let $\Gamma_n(A)$ be an approximation of a_n from above with $\Gamma_n(A) \leq a_n + 1/n$.

Exercise 7.3

The lower bounds follow from the lower bounds in the book for Ω_{D} and $\Omega_{\mathrm{B}} \cap \Omega_{\mathrm{SA}}$. For the upper bounds, we alter the solutions to the previous two exercises. Let $p_1, p_2, \ldots$ be an enumeration of the monic polynomials with rational coefficients and degree at least one. From Exercise 7.1, we know that there is Σ_1^A-tower for $\|p_j(A)\|$ for $A \in \Omega_{\mathrm{B}} \cap \Omega_f$. However, care must be taken since we are not assuming access to upper bounds for $\|A\|$. Hence, we run the Σ_1^A-tower with an approximation n for $\|A\|$. This leads to a Δ_2^A tower, $\{\tilde{\Gamma}_n\}$ such that

$$\lim_{n\to\infty} \tilde{\Gamma}_n(A, p_j) = \|p_j(A)\| \quad \forall A \in \Omega_{\mathrm{B}} \cap \Omega_f.$$

The following provides a Π_2^A-tower for $\{\Xi_{\mathrm{cap}}, \Omega_{\mathrm{B}} \cap \Omega_f, \mathbb{R}_{\geq 0}, \Lambda\}$:

$$\Gamma_{n_2,n_1}(A) = \min_{1\leq j\leq n_2} \tilde{\Gamma}_{n_1}(A, p_j)^{1/\deg(p_j)}.$$

The class Ω_{B} is similar but requires an additional limit to compute $\|p_j(A)\|$.

Exercise 7.4

Let $A \in \Omega_{\mathrm{B}} \cap \Omega_{\mathrm{SA}}$. Consider the sets

$$S_n(A) = \mathrm{Sp}(A) \cap (\mathrm{Sp}_{\mathrm{ess}}(A) + B_{1/n}(0)) \quad n \in \mathbb{N}.$$

The set $\mathrm{Sp}(A) \setminus (\mathrm{Sp}_{\mathrm{ess}}(A) + B_{1/n}(0))$ is empty or finite, and hence $\mathrm{cap}(\mathrm{Sp}(A)) = \mathrm{cap}(S_n(A))$. The capacity is right-continuous and, hence, $\mathrm{cap}(S_n(A)) \downarrow \mathrm{cap}(\mathrm{Sp}_{\mathrm{ess}}(A))$ as $n \to \infty$. The same argument works for the Lebesgue measure and deciding if the Lebesgue measure or capacity is zero. Clearly, the number of connected components of the essential spectrum can be less than that of the spectrum. By considering an operator with spectrum $\{0, 1, 1/2, 1/3, \ldots\}$, we see that the box-counting dimensions can differ. However, the Hausdorff dimension is invariant under countable perturbations and, hence, is the same for the spectrum and essential spectrum.

Exercise 7.5

We apply the proposition with the sequence $\delta_j = C\epsilon_{n_j}\log(1/\epsilon_{n_j})$ and $S = \mathrm{Sp}_+(\alpha, 1)$. Then $\mathrm{Sp}_+(p_{n_j}/q_{n_j}, 1) + B_\eta(0)$ can be covered by $\mathrm{Sp}_+(p_{n_j}/q_{n_j}, 1)$ and $4q_{n_j}$ closed intervals of length η. Let $\beta \in (1/2, 1)$, then $\delta_j \lesssim \epsilon_{n_j}^{\beta} \leq q_{n_j}^{-2\beta}$. It follows that

$$|S + B_{\delta_j}(0)| \leq |\mathrm{Sp}_+(p_{n_j}/q_{n_j}, 1) + B_{2\delta_j}(0)| \leq 8e/q_{n_j} + 8q_{n_j}\delta_j \leq \hat{C} q_{n_j}^{1-2\beta}$$

for some constant $\hat{C}$. In particular,

$$\log(|S + B_{\delta_j}(0)|) \leq \log(\hat{C}) + (2\beta - 1)\log(1/q_{n_j}), \quad \log(1/\delta_j) \geq \log(C') - 2\beta\log(1/q_{n_j})$$

for a constant C'. Hence,

$$1 - \limsup_{\delta\downarrow 0} \frac{\log(|S + B_\delta(0)|)}{\log(\delta)} = 1 + \liminf_{\delta\downarrow 0} \frac{\log(|S + B_\delta(0)|)}{\log(1/\delta)} \leq 1 - \frac{2\beta - 1}{2\beta} = \frac{1}{2\beta}.$$

We now take $\beta \uparrow 1$ to see that $\underline{\dim}_{\mathrm{B}}(\mathrm{Sp}_+(\alpha, 1)) \leq 1/2$.

For the bound on the upper box-counting dimension, we let $\delta_n = 13.2\sqrt{|\alpha - p_n/q_n|}$, then $S + B_{\delta_n}(0)$ covers $\mathrm{Sp}_+(p_n/q_n, 1)$ and, hence,

$$\log(|S + B_{\delta_n}(0)|) \geq \log(|\mathrm{Sp}_+(p_n/q_n, 1)|) \geq C' + \log(1/q_n), \quad \log(1/\delta_n) \leq C' + \frac{1}{2}\log(|\alpha - p_n/q_n|^{-1})$$

for a constant C'. Hence,

$$1 - \liminf_{\delta\downarrow 0} \frac{\log(|S + B_\delta(0)|)}{\log(\delta)} = 1 + \limsup_{\delta\downarrow 0} \frac{\log(|S + B_\delta(0)|)}{\log(1/\delta)} \geq 1 + \limsup_{n\to\infty} \frac{2\log(1/q_n)}{\log(|\alpha - p_n/q_n|^{-1})}.$$

Using the hint, we have, for every $\delta' > 0$,

$$\limsup_{n\to\infty} \frac{2\log(1/q_n)}{\log(|\alpha - p_n/q_n|^{-1})} \geq \frac{2\log(1/q_n)}{(\mu(\alpha) + \delta')\log(q_n)} = -\frac{2}{\mu(\alpha) + \delta'}.$$

Hence, the required result follows by taking $\delta' \downarrow 0$.

Exercise 7.6

We have

$$\lim_{\epsilon\downarrow 0}\Xi_{\mathrm{Lm},\epsilon}(A)=\Xi_{\mathrm{Lm}}(A)\quad \forall A\in\Omega_{\mathrm{B}}\cap\Omega_{\mathrm{SA}}.$$

Hence, the lower bounds for Ξ_{Lm} proven in the chapter immediately imply the lower bounds in the theorem we prove in this exercise. Once we have proven the upper bounds in the theorem, by taking $\epsilon\downarrow 0$, this limit also provides the upper bounds for Ξ_{Lm} for the classes $\Omega_{\mathrm{B}}\cap\Omega_f\cap\Omega_{\mathrm{SA}}$ and $\Omega_{\mathrm{B}}\cap\Omega_{\mathrm{SA}}$ (thus answering the final part of the exercise). Hence, it suffices to prove that $\{\Xi_{\mathrm{Lm},\epsilon},\Omega_{\mathrm{B}}\cap\Omega_f\cap\Omega_{\mathrm{SA}},\mathbb{R}_{\geq 0},\Lambda\}\in\Sigma_1^A$ and $\{\Xi_{\mathrm{Lm},\epsilon},\Omega_{\mathrm{B}}\cap\Omega_{\mathrm{SA}},\mathbb{R}_{\geq 0},\Lambda\}\in\Sigma_2^A$.

Let $A\in\Omega_{\mathrm{B}}\cap\Omega_f\cap\Omega_{\mathrm{SA}}$. We know from Chapter 3 that we may compute $F_n(z)$ in finitely many arithmetic operations and comparisons, so that $F_n(z)$ converges uniformly to $\|(A-zI)^{-1}\|^{-1}$ from above on compact subsets of $\mathbb{C}$. Set

$$E_n=\frac{1}{n}\mathbb{Z}\cap\{z\in\mathbb{C}:F_n(z)\leq\epsilon\}\cap[-n,n].$$

Clearly, we can compute E_n with finitely many arithmetic operations and comparisons and we set

$$\Gamma_n(A)=\left|\bigcup_{z\in E_n}D(z,\max\{0,\epsilon-F_n(z)\})\cap\mathbb{R}\right|.$$

Without loss of generality, we can assume $\Gamma_n(A)$ can be computed exactly using finitely many arithmetic operations and comparisons. Suppose that $F_n(z)<\epsilon$ and that $|w|<\epsilon-F_n(z)$. If $z\in\mathrm{Sp}(A)$ then clearly

$$\left\|(A-(z+w)I)^{-1}\right\|^{-1}\leq|w|<\epsilon-F_n(z)\leq\epsilon,$$

and this holds trivially if $z+w\in\mathrm{Sp}(A)$. So assume that neither of $z,z+w$ are in the spectrum. The resolvent identity yields

$$\left\|(A-(z+w)I)^{-1}\right\|\geq\left\|(A-zI)^{-1}\right\|-|w|\left\|(A-(z+w)I)^{-1}\right\|\left\|(A-zI)^{-1}\right\|,$$

which rearranges to

$$\left\|(A-(z+w)I)^{-1}\right\|^{-1}\leq\left\|(A-zI)^{-1}\right\|^{-1}+|w|<\epsilon.$$

It follows that $\cup_{z\in E_n}D(z,\max\{0,\epsilon-F_n(z)\})$ is in $\mathrm{Sp}_\epsilon(A)$ and, hence, that $\Gamma_n(A)\leq\Xi_{\mathrm{Lm},\epsilon}(A)$. Without loss of generality by taking successive maxima we can assume that $\Gamma_n(A)$ is increasing. Together, these facts will yield the Σ_1^A classification once convergence is shown. Using the uniform convergence of F_n and density of $\frac{1}{n}\mathbb{Z}\cap[-n,n]$, we see that pointwise convergence holds:

$$\chi_{\cup_{z\in E_n}D(z,\max\{0,\epsilon-F_n(z)\})\cap\mathbb{R}}\to\chi_{\{z:\|(A-zI)^{-1}\|^{-1}<\epsilon\}\cap\mathbb{R}},$$

where χ_E denotes the indicator function of a set E. It follows by the dominated convergence theorem that

$$\lim_{n\to\infty}\Gamma_n(A)=|\{z:\|(A-zI)^{-1}\|^{-1}<\epsilon\}\cap\mathbb{R}|=\Xi_{\mathrm{Lm},\epsilon}(A).$$

Here, we have used the fact that the set of points $z\in\mathbb{R}$ with $\|(A-zI)^{-1}\|^{-1}=\epsilon$ is countable and so has Lebesgue measure 0. For $A\in\Omega_{\mathrm{B}}\cap\Omega_{\mathrm{SA}}$, we simply replace F_{n_1} by suitable $1/n_2$-gridded approximations of γ_{n_2,n_1} (call them F_{n_2,n_1}) and set

$$\Gamma_{n_2,n_1}(A)=\left|\bigcup_{z\in E_{n_2}}D(z,\max\{0,\epsilon-F_{n_2,n_1}(z)\})\cap\mathbb{R}\right|.$$

Exercise 7.7

We may write

$$A-A_n=\int_{\mathrm{Sp}(A)}\left(\lambda-\frac{\lfloor\lambda\cdot n\rfloor}{n}\right)\mathrm{d}\mathcal{E}(\lambda).$$

It follows that for all $x\in\mathcal{H}$ with $\|x\|=1$,

$$\|(A-A_n)x\|^2=\int_{\mathrm{Sp}(A)}\left|\lambda-\frac{\lfloor\lambda\cdot n\rfloor}{n}\right|^2\mathrm{d}\mu_x(\lambda)\leq\frac{1}{n^2},$$

so that $\|A_n - A\| \leq 1/n$. However, $\mathrm{Sp}(A_n) \subset \frac{1}{n}\mathbb{Z}$ and, hence, $\Xi_{\mathrm{Lm}}(A_n) = 0$.

Suppose now that $\Xi_{\mathrm{Lm}}(A) = 0$ and A_n is a sequence of self-adjoint operators with $\lim_{n\to\infty} \|A - A_n\| = 0$. This implies that $\mathrm{Sp}(A_n) \to \mathrm{Sp}(A)$ since all our operators are normal. To prove that $|\mathrm{Sp}(A_n)| \to 0$, it is enough to prove that $|X_n| \downarrow 0$, where $X_n = \mathrm{Sp}(A) \cup (\cup_{m\geq n}\mathrm{Sp}(A_m))$. But X_n decreases to $\mathrm{Sp}(A)$ and is bounded in measure, so $|X_n| \downarrow 0$.

Now let A be a bounded self-adjoint operator, A_n be a sequence of self-adjoint operators with $\lim_{n\to\infty} \|A - A_n\| = 0$, and $\epsilon > 0$. Then given some $0 < \delta < \epsilon$ it holds for large n that $\mathrm{Sp}_{\epsilon-\delta}(A) \subset \mathrm{Sp}_{\epsilon}(A_n) \subset \mathrm{Sp}_{\epsilon+\delta}(A)$ and hence that

$$\limsup_{n\to\infty} \Xi_{\mathrm{Lm},\epsilon}(A_n) \leq \Xi_{\mathrm{Lm},\epsilon+\delta}(A), \quad \liminf_{n\to\infty} \Xi_{\mathrm{Lm},\epsilon}(A_n) \geq \Xi_{\mathrm{Lm},\epsilon-\delta}(A).$$

Now let $\delta \downarrow 0$ and use the fact that $\Xi_{\mathrm{Lm},\epsilon}$ is continuous in ϵ.

Exercise 7.8

Throughout this solution, we use $|\cdot|$ to denote the two-dimensional Lebesgue measure on $\mathbb{C} \cong \mathbb{R}^2$. We first prove that $\{\Xi^{\mathbb{C}}_{\mathrm{Lm}}, \Omega_{\mathrm{B}} \cap \Omega_f, \mathbb{R}_{\geq 0}, \Lambda\} \in \Pi^A_2$. For $A \in \Omega_{\mathrm{B}} \cap \Omega_f$, we will estimate $\Xi^{\mathbb{C}}_{\mathrm{Lm}}(A)$ by estimating the Lebesgue measure of the resolvent set on the closed square $[-C, C]^2$, where $\|A\| \leq C$. We do not assume C is known. For $n_1, n_2 \in \mathbb{N}$, let

$$\mathrm{Grid}(n_1, n_2) = \left(\frac{1}{2^{n_2}}\mathbb{Z} + \frac{1}{2^{n_2}}\mathrm{i}\mathbb{Z}\right) \cap [-n_1, n_1]^2.$$

We know from Chapter 3 that we may compute $F_n(z)$ in finitely many arithmetic operations and comparisons, so that $F_n(z)$ converges uniformly to $\|(A - zI)^{-1}\|^{-1}$ from above on compact subsets of $\mathbb{C}$. Letting $B(x, r)$ and $D(x, r)$ denote the closed and open balls of radius r around x, respectively (we set $D(x, 0) = \emptyset$), in $\mathbb{C}$, we define

$$U(n_1, n_2, A) = [-n_1, n_1] \times [-n_1, n_1] \cap (\cup_{z\in\mathrm{Grid}(n_1,n_2)} B(z, F_{n_1}(z))).$$

Note that $|U(n_1, n_2, A)|$ can be computed up to any desired precision using finitely many arithmetic operations and comparisons. Hence, we can define

$$\Gamma_{n_2,n_1}(A) = 4n_1^2 - |U(n_1, n_2, A)|,$$

where, without loss of generality, we assume that we have computed the exact value of the Lebesgue measure of $U(n_1, n_2, A)$ (since we can absorb this error in the first limit). Clearly, Γ_{n_2,n_1} are arithmetical algorithms using Λ, so we must prove convergence. There exists a compact set K such that $\|(A - zI)^{-1}\|^{-1} > 1$ on $\mathbb{C} \setminus K$ and, without loss of generality by making C larger, we can take $C \in \mathbb{N}$ and $K = [-C, C]^2$. Since $F_n(z) \geq \|(A - zI)^{-1}\|^{-1}$, if $n_1 \geq C$, then

$$U(n_1, n_2, A) = ([-C, C]^2 \cap (\cup_{z\in\mathrm{Grid}(n_1,n_2)} B(z, F_{n_1}(z)))) \cup ([-n_1, n_1]^2 \setminus [-C, C]^2).$$

It follows that for large n_1

$$\Gamma_{n_2,n_1}(A) = 4C^2 - \left|[-C, C]^2 \cap (\cup_{z\in\mathrm{Grid}(n_1,n_2)} B(z, F_{n_1}(z)))\right|.$$

As $n_1 \to \infty$, $[-C, C]^2 \cap (\cup_{z\in\mathrm{Grid}(n_1,n_2)} B(z, F_{n_1}(z)))$ converges to the closed set

$$K(n_2, A) = [-C, C]^2 \cap (\cup_{z\in\mathrm{Grid}(C,n_2)} B(z, \|(A - zI)^{-1}\|^{-1}))$$

from above and, hence,

$$\lim_{n_1\to\infty} \Gamma_{n_2,n_1}(A) = 4C^2 - |K(n_2, A)|$$

from below. Consider the relatively open set

$$V(n_2, A) = [-C, C]^2 \cap (\cup_{z\in\mathrm{Grid}(C,n_2)} D(z, \|(A - zI)^{-1}\|^{-1})).$$

Clearly, $|K(n_2, A)| = |V(n_2, A)|$ since the sets differ by a finite collection of circular arcs or points (recall we defined the open ball of radius zero to be the empty set). Hence, we must show that $\lim_{n_2\to\infty} |V(n_2, A)| = |\rho_C(A)|$, where $\rho_C(A) = [-C, C]^2 \setminus \mathrm{Sp}(A)$. For $z \in \rho_C(A)$, $\mathrm{dist}(z, \mathrm{Sp}(A)) \geq \|(A - zI)^{-1}\|^{-1}$ and, hence, we get $V(n_2, A) \subset \rho_C(A)$. Since $\rho_C(A)$ is relatively open, a simple density argument using the continuity of $\|(A - zI)^{-1}\|^{-1}$ yields $V(n_2, A) \uparrow \rho_C(A)$ as $n_2 \to \infty$ since the grid refines itself. So we get $|V(n_2, A)| \uparrow |\rho_C(A)|$. This proves the convergence and also shows that $\Gamma_{n_2}(A) \downarrow \Xi^{\mathbb{C}}_{\mathrm{Lm}}(A)$, thus yielding the Π^A_2 classification.

The proof that $\{\Xi_{\mathrm{Lm}}^{\mathbb{C}}, \Omega_{\mathrm{B}} \cap \Omega_f, \mathbb{R}_{\geq 0}, \Lambda\} \notin \Delta_2^G$ is similar to the proof that $\{\Xi_{\mathrm{Lm}}, \Omega_{\mathrm{D}}, \mathbb{R}_{\geq 0}, \Lambda\} \notin \Delta_2^G$. However, we pick a sequence that is dense in some set that has positive (two-dimensional) Lebesgue measure, such as a square or disk.

Next, we prove that $\{\Xi_{\mathrm{Lm}}^{\mathbb{C}}, \Omega_{\mathrm{B}}, \mathbb{R}_{\geq 0}, \Lambda\} \in \Pi_3^A$. Recall the functions

$$\gamma_{n_2,n_1}(z, A) = \min\left\{\sigma_{\inf}(\mathcal{P}_{n_1}(A - zI)\mathcal{P}_{n_2}^*), \sigma_{\inf}(\mathcal{P}_{n_1}(A^* - \overline{z}I)\mathcal{P}_{n_2}^*)\right\},$$
$$\gamma_{n_2}(z, A) = \min\left\{\sigma_{\inf}((A - zI)\mathcal{P}_{n_2}^*), \sigma_{\inf}((A^* - \overline{z}I)\mathcal{P}_{n_2}^*)\right\}.$$

In the same manner as before, define

$$U(n_1, n_2, n_3, A) = [-n_2, n_2]^2 \cap (\cup_{z\in\mathrm{Grid}(n_2,n_3)} B(z, \hat{\gamma}_{n_2,n_1}(z; A))), \quad \Gamma_{n_3,n_2,n_1}(A) = (2n_2)^2 - |U(n_1, n_2, n_3, A)|,$$

where $\hat{\gamma}_{n_2,n_1}$ is a computed approximation of γ_{n_2,n_1} accurate to $1/n_1$. Note that $\hat{\gamma}_{n_2,n_1}$ converges locally uniformly to γ_{n_2} as $n_1 \to \infty$ and we may use γ_{n_2} instead of F_{n_1}. The arguments in the proof of $\{\Xi_{\mathrm{Lm}}, \Omega_{\mathrm{D}}, \mathbb{R}_{\geq 0}, \Lambda\} \notin \Delta_2^G$ show that $\{\Gamma_{n_3,n_2,n_1}\}$ is a Π_3^A-tower for $\{\Xi_{\mathrm{Lm}}^{\mathbb{C}}, \Omega_{\mathrm{B}} \cap \Omega_f, \mathbb{R}_{\geq 0}, \Lambda\}$.

Again, the proof that $\{\Xi_{\mathrm{Lm}}^{\mathbb{C}}, \Omega_{\mathrm{B}}, \mathbb{R}_{\geq 0}, \Lambda\} \notin \Delta_3^G$ is a simple adaptation of the proof that $\{\Xi_{\mathrm{Lm}}, \Omega_{\mathrm{B}} \cap \Omega_{\mathrm{SA}}, \mathbb{R}_{\geq 0}, \Lambda\} \notin \Delta_3^G$. We can also ensure that the operators in question are normal, and, hence, $\{\Xi_{\mathrm{Lm}}^{\mathbb{C}}, \Omega_{\mathrm{B}} \cap \Omega_g, \mathbb{R}_{\geq 0}, \Lambda\} \notin \Delta_3^G$. Since $\Omega_{\mathrm{B}} \cap \Omega_g \subset \Omega_{\mathrm{B}}$, we also have $\{\Xi_{\mathrm{Lm}}^{\mathbb{C}}, \Omega_{\mathrm{B}} \cap \Omega_g, \mathbb{R}_{\geq 0}, \Lambda\} \in \Pi_3^A$.

For the final part, the proof that $\{\Xi_{\mathrm{Lm}}^{\mathrm{dec}}, \Omega_{\mathrm{B}} \cap \Omega_{\mathrm{SA}}, \mathcal{M}_{\mathrm{dec}}, \Lambda\} \notin \Delta_4^G$ can be extended to two-dimensional Lebesgue measure and the class Ω_{B}. To prove that the problem lies in Π_4^A, we take the Π_3^A-tower for $\{\Xi_{\mathrm{Lm}}^{\mathbb{C}}, \Omega_{\mathrm{B}}, \mathbb{R}_{\geq 0}, \Lambda\}$, $\{\Gamma_{n_3,n_2,n_1}\}$. Without loss of generality, Γ_{n_3,n_2,n_1} is decreasing in n_1, increasing in n_2, decreasing in n_3. Now let

$$\hat{\Gamma}_{n_4,n_3,n_2,n_1}(A) = \chi_{[0,1/n_4]}(\Gamma_{n_3,n_2,n_1}(A)).$$

Note that $\chi_{[0,1/n_4]}$ is left continuous on $[0, \infty)$ with right limits. Hence, by the assumed monotonicity

$$\lim_{n_1\to\infty} \hat{\Gamma}_{n_4,n_3,n_2,n_1}(A) = \chi_{[0,1/n_4]}(\Gamma_{n_3,n_2}(A)\pm),$$

where $\pm$ denotes one of the right or left limits (it is possible to have either). Passing to the next two limits,

$$\lim_{n_3\to\infty} \lim_{n_2\to\infty} \lim_{n_1\to\infty} \hat{\Gamma}_{n_4,n_3,n_2,n_1}(A) = \chi_{[0,1/n_4]}(\Xi_{\mathrm{Lm}}^{\mathbb{C}}(A)\pm).$$

If $\Xi_{\mathrm{Lm}}^{\mathbb{C}}(A) = 0$, then the quantity on the right-hand side is 1. If $\Xi_{\mathrm{Lm}}^{\mathbb{C}}(A) > 0$, the quantity is 0 for sufficiently large n_4. Moreover, if the quantity is 0, then $\Xi_{\mathrm{Lm}}^{\mathbb{C}}(A) > 0$. Hence, we obtain Π_4^A convergence.

Exercise 7.9

Let K be a compact countable set and μ a Borel probability measure supported on K. Since K has atoms, we must have $E(\mu) = +\infty$. Since μ was arbitrary, $\mathrm{cap}(K) = 0$. By a rescaling of the example with the Joukowsky map in the book, the capacity of $[0, 1]$ is nonzero. Recall the decision problems

$$\Xi_{\mathrm{cap}}^{\mathrm{dec}}(A) = \begin{cases} 1, & \text{if } \Xi_{\mathrm{cap}}(A) = 0, \\ 0, & \text{otherwise,} \end{cases} \qquad \Xi_{\mathrm{Lm}}^{\mathrm{dec}}(A) = \begin{cases} 1, & \text{if } \Xi_{\mathrm{Lm}}(A) = 0, \\ 0, & \text{otherwise,} \end{cases}$$

which map to the metric space $\mathcal{M}_{\mathrm{dec}}$. Consider first the class Ω_{D}. Using the fact that the capacity of a compact countable set is zero, the proof that $\{\Xi_{\mathrm{Lm}}^{\mathrm{dec}}, \Omega_{\mathrm{D}}, \mathcal{M}_{\mathrm{dec}}, \Lambda\} \notin \Delta_3^G$ carries over directly to show that $\{\Xi_{\mathrm{cap}}^{\mathrm{dec}}, \Omega_{\mathrm{D}}, \mathcal{M}_{\mathrm{dec}}, \Lambda\} \notin \Delta_3^G$. Moreover, since the capacity of $[0, 1]$ is nonzero, the proof that $\{\Xi_{\mathrm{Lm}}, \Omega_{\mathrm{D}}, \mathbb{R}_{\geq 0}, \Lambda\} \notin \Delta_2^G$ carries over to show that $\{\Xi_{\mathrm{cap}}, \Omega_{\mathrm{D}}, \mathbb{R}_{\geq 0}, \Lambda\} \notin \Delta_2^G$. In a similar fashion, the proofs that $\{\Xi_{\mathrm{Lm}}^{\mathrm{dec}}, \Omega_{\mathrm{B}} \cap \Omega_{\mathrm{SA}}, \mathcal{M}_{\mathrm{dec}}, \Lambda\} \notin \Delta_4^G$ and $\{\Xi_{\mathrm{Lm}}, \Omega_{\mathrm{B}} \cap \Omega_{\mathrm{SA}}, \mathbb{R}_{\geq 0}, \Lambda\} \notin \Delta_3^G$ show that $\{\Xi_{\mathrm{cap}}^{\mathrm{dec}}, \Omega_{\mathrm{B}} \cap \Omega_{\mathrm{SA}}, \mathcal{M}_{\mathrm{dec}}, \Lambda\} \notin \Delta_4^G$ and $\{\Xi_{\mathrm{cap}}, \Omega_{\mathrm{B}} \cap \Omega_{\mathrm{SA}}, \mathbb{R}_{\geq 0}, \Lambda\} \notin \Delta_3^G$, respectively.

Exercise 7.10

Let $K \subset \mathbb{R}$ be compact and countable. We may list the elements of K as $\{x_j\}$, which may be finite or countably infinite. Let $\epsilon > 0$, $d \in (0, 1]$, and $\{a_n\} \subset \mathbb{R}_{>0}$ be a sequence with $\sum_{n=1}^{\infty} a_n^d \leq \epsilon$. For $k \in \mathbb{N}$, we may choose $U_j^{(k)} \in \cup_{l\geq k}\rho_l$ with

$|U_j^{(k)}| \leq a_j$ and $x_j \in \mathrm{int}(U_j^{(k)})$. Since K is compact, there exists a finite subset I such that $\cup_{j\in I}\mathrm{int}(U_j^{(k)})$ covers K and, hence,

$$H_k^d(K) \leq \sum_{j\in I} a_j^d \leq \epsilon.$$

Since $\epsilon > 0$ was arbitrary, $H_k^d(K) = 0$. It follows that $\dim_{\mathrm{H}}(K) = 0$.

To prove that a non-degenerate closed interval has Hausdorff dimension equal to 1, we let $K = [0, 1]$ and other such intervals are similar. Using the fact that $x \mapsto x^d$ is concave for $d \in (0, 1]$ and Jensen's inequality, we have that

$$H_k^d(K) = \sum_{j=1}^{2^k} 2^{-kd} = 2^{k(1-d)}.$$

It follows that $H^d(K) = +\infty$ if $d < 1$ and $H^1(K) = 1$, i.e., $\dim_{\mathrm{H}}(K) = 1$.

The proofs of lower bounds for $\Xi_{\mathrm{Lm}}^{\mathrm{dec}}$ now adapt trivially to Ξ_{H}.

Exercise 7.11

We first adapt the limit-computable covers to unbounded self-adjoint operators. Suppose that **(S2)** holds for a class Ω of (possibly unbounded) self-adjoint operators. For a compact (non-degenerate) interval $[a, b]$ and $n_2 \in \mathbb{N}$, let

$$G_{1/n_2}^{[a,b]} = \left\{ a + (k + 1/2)\frac{(b-a)}{n_2} : k = 0, \dots, n_2 - 1 \right\}.$$

Recall that given $n_1, n_2 \in \mathbb{N}$ and $A \in \Omega$, we define the function

$$\widehat{\Phi}_{n_1}(n_2; z, A) = \sup\left\{ \frac{k}{n_2} : k \in \mathbb{Z}_{\geq 0}, \frac{k}{n_2} \leq \Phi_{n_1}(z, A) \right\}.$$

We then define:

$$\Gamma_{n_2,n_1}^{\mathrm{Sp}}(a, b; A) = [a, b] \setminus \bigcup_{w\in G_{1/n_2}^{[a,b]}} D\left(w, \widehat{\Phi}_{n_1}(n_2; w, A)\right),$$

where $D(x, r)$ is the open ball of radius r centred at x and is taken to be the empty set if $r = 0$. We now have the following lemma.

Lemma 7.1. *For each fixed n_2, a, b, and A, $\Gamma_{n_2,n_1}^{\mathrm{Sp}}(a, b; A)$ is eventually constant for large n_1. Moreover, the limit set $\Gamma_{n_2}^{\mathrm{Sp}}(a, b; A) = \lim_{n_1\to\infty} \Gamma_{n_2,n_1}^{\mathrm{Sp}}(a, b; A)$ satisfies*

$$\mathrm{Sp}(A) \cap [a, b] \subset \Gamma_{n_2}^{\mathrm{Sp}}(a, b; A) \subset \mathrm{Sp}(A) \cap [a, b] + B_{(b+1-a)/n_2}(0).$$

Proof. The convergence of the functions $\{\Phi_n\}$ from above implies that

$$\lim_{n_1\to\infty} \widehat{\Phi}_{n_1}(n_2; z, A) = \widehat{\Phi}(n_2; z, A) = \sup\left\{ \frac{k}{n_2} : k \in \mathbb{Z}_{\geq 0}, \frac{k}{n_2} \leq \mathrm{dist}(z, \mathrm{Sp}(A)) \right\}.$$

Moreover, for each fixed n_2, z and A, $\widehat{\Phi}_{n_1}(n_2; z, A)$ is eventually constant for large n_1. It follows that $\Gamma_{n_2,n_1}^{\mathrm{Sp}}(a, b; A)$ is constant for large n_1 with

$$\Gamma_{n_2}^{\mathrm{Sp}}(a, b; A) = \lim_{n_1\to\infty} \Gamma_{n_2,n_1}^{\mathrm{Sp}}(a, b; A) = [a, b] \setminus \bigcup_{w\in G_{1/n_2}^{[a,b]}} D\left(w, \widehat{\Phi}(n_2; w, A)\right).$$

Now suppose that $z \in \mathrm{Sp}(A) \cap [a, b]$ and $w \in G_{1/n_2}^{[a,b]}$. Then

$$|z - w| \geq \mathrm{dist}(w, \mathrm{Sp}(A)) \geq \widehat{\Phi}(n_2; w, A)$$

and, hence,

$$z \notin \bigcup_{w \in G_{1/n_2}^{[a,b]}} D\left(w, \widehat{\Phi}(n_2; w, A)\right).$$

It follows that $\mathrm{Sp}(A) \cap [a,b] \subset \Gamma_{n_2}^{\mathrm{Sp}}(a,b;A)$. On the other hand, if $z \in \Gamma_{n_2}^{\mathrm{Sp}}(a,b;A)$, let w be a nearest point in $G_{1/n_2}^{[a,b]}$ to z. Then $\widehat{\Phi}(n_2; w, A) \leq |w-z| \leq (b-a)/(2n_2)$, which also implies that the points in $\mathrm{Sp}(A)$ nearest to w must also be in $[a,b]$. Hence,

$$\mathrm{dist}(z, \mathrm{Sp}(A) \cap [a,b]) \leq |w-z| + \mathrm{dist}(w, \mathrm{Sp}(A) \cap [a,b]) = |w-z| + \mathrm{dist}(w, \mathrm{Sp}(A)) \leq |w-z| + \widehat{\Phi}(n_2; w, A) + \frac{1}{n_2} \leq \frac{b+1-a}{n_2}.$$

The lemma now follows. □

With this lemma in hand, we now deal with the upper bounds.

Connected components using (S2): Let $\#(X)$ be the (potentially infinite) number of connected components of a set $X \subset \mathbb{R}$. For each $[a,b]$, we can use Lemma 7.1 to construct a Σ_2^A-tower, $\{\Gamma_{n_2,n_1}^{[a,b]}\}$, for $\#(\mathrm{Sp}(A) \cap [a,b])$. A little care is needed when $\mathrm{Sp}(A) \cap [a,b] = \emptyset$, but in this case we can use the fact that $\Gamma_{n_2}^{\mathrm{Sp}}(a,b;A)$ is empty if and only if $\mathrm{Sp}(A) \cap [a,b] = \emptyset$. We then set

$$\Gamma_{n_2,n_1}(A) = \max_{1 \leq n \leq n_2} \Gamma_{n_2,n_1}^{[-n,n]}(A).$$

Then

$$\lim_{n_1 \to \infty} \Gamma_{n_2,n_1}(A) = \Gamma_{n_2}(A) = \max_{1 \leq n \leq n_2} \Gamma_{n_2}^{[-n,n]}(A) \leq \max_{1 \leq n \leq n_2} \#(\mathrm{Sp}(A) \cap [-n,n]) \leq \#(\mathrm{Sp}(A)).$$

Moreover, for every $N \in \mathbb{N}$, we have

$$\liminf_{n_2 \to \infty} \Gamma_{n_2}(A) \geq \lim_{n_2 \to \infty} \Gamma_{n_2}^{[-N,N]}(A) = \#(\mathrm{Sp}(A) \cap [-N,N]).$$

Taking $N \to \infty$ on the right-hand side, we see that $\lim_{n_2 \to \infty} \Gamma_{n_2}(A) = \#(\mathrm{Sp}(A))$, and, hence, $\{\Xi_{\mathrm{cc}}, \Omega, \mathbb{N} \cup \{+\infty\}, \Lambda\} \in \Sigma_2^A$. To deal with the decision problem $\Xi_{\mathrm{cc}}^{\mathrm{dec}}$, we set

$$\hat{\Gamma}_{n_3,n_2,n_1}(A) = \begin{cases} 1, & \text{if } \Gamma_{n_2,n_1}(A) > n_3, \\ 0, & \text{otherwise,} \end{cases}$$

which provides a Π_3^A-tower.

Lebesgue measure and capacity using (S2): In the same way, we obtain a Π_2^A-tower, $\{\Gamma_{n_2,n_1}^{[a,b]}\}$, for $|\mathrm{Sp}(A) \cap [a,b]|$. We now take $\Gamma_{n_3,n_2,n_1}(A) = \Gamma_{n_2,n_1}^{[-n_3,n_3]}(A)$, to see that $\{\Xi_{\mathrm{Lm}}, \Omega, \mathbb{R}_{\geq 0} \cup \{+\infty\}, \Lambda\} \in \Sigma_3^A$. For the decision problem $\Xi_{\mathrm{Lm}}^{\mathrm{dec}}$, we set

$$\hat{\Gamma}_{n_3,n_2,n_1}(A) = \begin{cases} 1, & \text{if } \min\limits_{1 \leq n \leq n_2} \Gamma_{n,n_1}^{[-n_3,n_3]}(A) < 1/n_3, \\ 0, & \text{otherwise.} \end{cases}$$

We then have

$$\lim_{n_1 \to \infty} \hat{\Gamma}_{n_3,n_2,n_1}(A) = \hat{\Gamma}_{n_3,n_2}(A) = \begin{cases} 1, & \text{if } \min\limits_{1 \leq n \leq n_2} \Gamma_n^{[-n_3,n_3]}(A) < 1/n_3, \\ 0, & \text{otherwise,} \end{cases}$$

$$\lim_{n_2 \to \infty} \hat{\Gamma}_{n_3,n_2}(A) = \hat{\Gamma}_{n_3}(A) = \begin{cases} 1, & \text{if } |\mathrm{Sp}(A) \cap [-n_3,n_3]| < 1/n_3, \\ 0, & \text{otherwise,} \end{cases}$$

which provides a Π_3^A-tower. The capacity is analogous, and so $\{\Xi_{\mathrm{cap}}, \Omega, \mathbb{R}_{\geq 0} \cup \{+\infty\}, \Lambda\} \in \Sigma_3^A$ and $\{\Xi_{\mathrm{cap}}^{\mathrm{dec}}, \Omega, \mathcal{M}_{\mathrm{dec}}, \Lambda\} \in \Pi_3^A$.

Hausdorff dimension using (S2): For each $[a,b]$, we can use Lemma 7.1 to construct a Σ_3^A-tower, $\{\Gamma_{n_3,n_2,n_1}^{[a,b]}\}$, for $\dim_{\mathrm{H}}(\mathrm{Sp}(A) \cap [a,b])$. We then set

$$\Gamma_{n_3,n_2,n_1}(A) = \max_{1 \leq n \leq n_3} \Gamma_{n_3,n_2,n_1}^{[-n,n]}(A).$$

Arguing as we did for capacity, this provides a Σ_3^A-tower for $\{\Xi_{\mathrm{H}}, \Omega, [0,1], \Lambda\}$.

Upper bounds using (S3): We first provide an easy generalisation of the double-limit covers, restricted to intervals $[a, b]$. Define

$$\Gamma^{\mathrm{Sp}}_{n_3,n_2,n_1}(a, b; A) = [a, b] \setminus \bigcup_{w \in G^{[a,b]}_{1/n_3}} D\left(w, \widehat{\Phi}_{n_2,n_1}(n_3; w, A)\right).$$

Arguing as in the book, we see that

$$\lim_{n_1\to\infty} \widehat{\Phi}_{n_2,n_1}(n_3; z, A) = \sup\left\{\frac{k}{n_3} : k \in \mathbb{Z}_{\geq 0}, \frac{k}{n_3} < \Phi_{n_2}(z, A) + \frac{1}{n_2}\right\},$$

where the limit is obtained for sufficiently large n_1. The $1/n_2$ is to ensure that $\Phi_{n_2}(z, A) + 1/n_2 > \mathrm{dist}(z, \mathrm{Sp}(A))$ so that

$$\lim_{n_2\to\infty} \lim_{n_1\to\infty} \widehat{\Phi}_{n_2,n_1}(n_3; z, A) = \sup\left\{\frac{k}{n_3} : k \in \mathbb{Z}_{\geq 0}, \frac{k}{n_3} \leq \mathrm{dist}(z, \mathrm{Sp}(A))\right\}.$$

This second limit is obtained for sufficiently large n_2. Hence, we have shown the following lemma.

Lemma 7.2. *For each fixed n_3, n_2, a, b, and A, $\Gamma^{\mathrm{Sp}}_{n_3,n_2,n_1}(a, b; A)$ is eventually constant for large n_1. Moreover, the limit set $\lim_{n_1\to\infty} \Gamma^{\mathrm{Sp}}_{n_3,n_2,n_1}(a, b; A)$ is eventually constant for large n_2, with*

$$\mathrm{Sp}(A) \cap [a, b] \subset \lim_{n_2\to\infty} \lim_{n_1\to\infty} \Gamma^{\mathrm{Sp}}_{n_3,n_2,n_1}(a, b; A) \subset \mathrm{Sp}(A) \cap [a, b] + B_{(b+1-a)/n_3}(0).$$

The desired upper bounds now follow by arguing as we did for **(S2)** but with the additional limit.

Lower bounds: We prove the lower bounds for Ξ_{Lm} and the lower bounds for Ξ_{cap} are entirely analogous.

Let Ω' denote the collection of all infinite matrices $a = \{a_{i,j}\}_{i,j\in\mathbb{N}}$ with entries $a_{i,j} \in \{0, 1\}$. Recall the problem function

$$\Xi_{2,P}(\{a_{i,j}\}) = \begin{cases} 1, & \text{if } \{a_{i,j}\} \text{ has a column containing infinitely many 1's,} \\ 0, & \text{otherwise.} \end{cases}$$

We showed in Chapter 2 that $\{\Xi_{2,P}, \Omega', [0, 1], \Lambda'\} \notin \Delta^G_3$, where Λ' is the set of component-wise evaluations of $\{a_{i,j}\}$. Suppose for a contradiction that $\{\Gamma_{n_2,n_1}\}$ is a Δ^G_3-tower of algorithms for $\{\Xi_{\mathrm{Lm}}, \Omega_f \cap \Omega_{\mathrm{SA}}, \mathbb{R}_{\geq 0} \cup \{+\infty\}, \Lambda\}$. Let $\{b_j\}_{j=0}^{\infty}$ be a sequence of distinct rational numbers in $(0, 1)$ that are dense in $[0, 1]$. Given $\{c_i\}_{i\in\mathbb{N}} \subset \{0, 1\}$, we let

$$\beta_j = \beta_j(\{c_i\}) = \sum_{i=1}^{j} c_i, \quad j \in \mathbb{N}.$$

Note that $\{b_0\} \cup \{b_{\beta_j} : j \in \mathbb{N}\} = \{b_j\}_{j=0}^{\sum_{i=1}^{\infty} c_i}$. Define the diagonal operator

$$C(\{c_i\}) = \mathrm{diag}(b_0, b_{\beta_1}, b_{\beta_2}, \ldots).$$

Then $\mathrm{Sp}(C(\{c_i\})) = [0, 1]$ if $\{c_i\}_{i\in\mathbb{N}}$ has infinitely many 1s, otherwise it is a finite subset of $(0, 1)$. Define the operator

$$A(a) = \bigoplus_{j=1}^{\infty} \left[C(\{a_{i,j}\}_{i\in\mathbb{N}}) + jI\right],$$

which can be realised as an unbounded diagonal operator on $\ell^2(\mathbb{N})$. The spectrum of $A(a)$ is the union of the intervals $[j, j + 1]$ over the columns $\{a_{i,j}\}_{i\in\mathbb{N}}$ with infinitely many 1s, together with a countable set. Each diagonal entry of $A(a)$ can be computed by evaluating finitely many of the entries of $\{a_{i,j}\}$. It follows that $a \mapsto \min\{\Gamma_{n_2,n_1}(A(a)), 1\}$ provides a Δ^G_3-tower of algorithms for the computational problem $\{\Xi_{2,P}, \Omega', [0, 1], \Lambda'\}$, the required contradiction.

Now let Ω' denote the collection of all infinite arrays $a = \{a_{m,i,j}\}_{m,i,j\in\mathbb{N}}$ with entries $a_{m,i,j} \in \{0, 1\}$. Recall the problem function

$$\Xi_{3,Q}(\{a_{m,i,j}\}) = \begin{cases} 1, & \text{if } \forall m,\ \sum_i a_{m,i,j} = \infty \text{ for all but finitely many } j, \\ 0, & \text{otherwise.} \end{cases}$$

In Chapter 2, we showed that $\{\Xi_{3,Q}, \Omega', [0,1], \Lambda'\} \notin \Delta_4^G$, where Λ' is the set of component-wise evaluations of $\{a_{m,i,j}\}$. Let $\{z_s\}_{s\in\mathbb{N}} \subset [-1,1]$ be a sequence of rational numbers that is dense in $[-1,1]$. For each $k \in \mathbb{N}$, define the symmetric block matrix

$$B(z_1,\ldots,z_k) = \left(\begin{array}{ccc|ccc} z_1 & & & \sqrt{1-z_1^2} & & \\ & \ddots & & & \ddots & \\ & & z_k & & & \sqrt{1-z_k^2} \\ \hline \sqrt{1-z_1^2} & & & -z_1 & & \\ & \ddots & & & \ddots & \\ & & \sqrt{1-z_k^2} & & & -z_k \end{array}\right), \tag{25}$$

where we take the positive square root for each $\sqrt{1-z_j^2}$. By a unitary change of basis, the above matrix is equivalent to a block diagonal matrix with blocks

$$\begin{pmatrix} z_j & \sqrt{1-z_j^2} \\ \sqrt{1-z_j^2} & -z_j \end{pmatrix}.$$

Hence, $B(z_1,\ldots,z_k)$ has eigenvalues ± 1, each repeated k times. Given $\{c_i\}_{i\in\mathbb{N}} \subset \{0,1\}$ and $j \in \mathbb{N}$, we define an operator $C^{(j)}(\{c_i\})$ in block matrix form as follows:

$$C^{(j)}(\{c_i\}) = \begin{pmatrix} \boldsymbol{C}_{1,1} & \boldsymbol{C}_{1,2} & \boldsymbol{C}_{1,3} & \cdots \\ \boldsymbol{C}_{2,1} & \boldsymbol{C}_{2,2} & \boldsymbol{C}_{2,3} & \cdots \\ \boldsymbol{C}_{3,1} & \boldsymbol{C}_{3,2} & \boldsymbol{C}_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where each block $\boldsymbol{C}_{k,l} \in \mathbb{C}^{j\times j}$. We first set $\boldsymbol{C}_{1,1} = \mathrm{diag}(z_1,\ldots,z_j)$. Then for all $k \in \mathbb{N}$, if $c_k = 1$, we let $r_k = \max\{l : c_l = 1, l < k\}$, where we take $r_k = 0$ if $\{l : c_l = 1, l < k\} = \emptyset$, and define

$$\boldsymbol{C}_{2k,2k} = \mathrm{diag}\left(-z_1,\ldots,-z_j\right), \quad \boldsymbol{C}_{2k+1,2k+1} = \mathrm{diag}\left(z_1,\ldots,z_j\right), \quad \boldsymbol{C}_{2k,2r_k+1} = \boldsymbol{C}_{2r_k+1,2k} = \mathrm{diag}\left(\sqrt{1-z_1^2},\ldots,\sqrt{1-z_j^2}\right).$$

If $c_k = 0$, we set

$$\boldsymbol{C}_{2k,2k} = \boldsymbol{C}_{2k+1,2k+1} = \mathrm{diag}(1,\ldots,1).$$

We set all of the other blocks in $C^{(j)}(\{c_i\})$ to zero. Each matrix entry of $C^{(j)}(\{c_i\})$ can be computed using finitely many evaluations of the sequence $\{c_i\}_{i\in\mathbb{N}}$. If the sequence $\{c_i\}_{i\in\mathbb{N}}$ has infinitely many 1s, then $C^{(j)}(\{c_i\})$ is unitarily equivalent to a direct sum of infinitely many $B(z_1,\ldots,z_j)$ together with the identity operator acting on some subspace. On the other hand, if $\{c_i\}_{i\in\mathbb{N}}$ has finitely many 1s, then $C^{(j)}(\{c_i\})$ is unitarily equivalent to a direct sum of a finite number of $B(z_1,\ldots,z_j)$, the operator $\mathrm{diag}(z_1,\ldots,z_j)$ and the identity operator acting on some subspace. More precisely, we have

$$\mathrm{Sp}(C^{(j)}(\{c_i\})) = \begin{cases} \left\{z_1,\ldots,z_j\right\} \cup \{1\}, & \text{if } \sum_i c_i = 0, \\ \left\{z_1,\ldots,z_j\right\} \cup \{-1,1\}, & \text{if } 0 < \sum_i c_i < \infty, \\ \{-1,1\}, & \text{if } \sum_i c_i = \infty. \end{cases}$$

Given $a = \{a_{m,i,j}\}_{m,i,j\in\mathbb{N}} \in \Omega'$ and $m \in \mathbb{N}$, we set

$$A_m(a) = \bigoplus_{j=1}^{\infty} C^{(j)}(\{a_{m,i,j}\}_{i\in\mathbb{N}}) \oplus \mathrm{diag}(-1, 1/m).$$

Recall the decision problem $\Xi_{2,Q}$, which tests whether a given matrix has only finitely many columns with finitely many 1s. For a fixed $m \in \mathbb{N}$, if $\Xi_{2,Q}(\{a_{m,i,j}\}_{i,j\in\mathbb{N}}) = 1$, then $\mathrm{Sp}(A_m(a))$ is a finite subset of $[-1,1]$. On the other hand, if $\Xi_{2,Q}(\{a_{m,i,j}\}_{i,j\in\mathbb{N}}) = 0$, then $\{z_1,\ldots,z_{j_l}\} \subset \mathrm{Sp}(A_m(a))$ for an increasing sequence $\{j_l\}_{l=1}^{\infty}$. Since $\{z_s\}_{s\in\mathbb{N}}$ is a dense subset of $[-1,1]$, $\mathrm{Sp}(A_m(a)) = [-1,1]$ in this case. We now define

$$A(a) = \bigoplus_{m=1}^{\infty} \left[A_m(a) + 2mI\right].$$

If $\Xi_{3,Q}(\{a_{m,i,j}\}_{m,i,j\in\mathbb{N}}) = 1$, then $\Xi_{2,Q}(\{a_{m,i,j}\}_{i,j\in\mathbb{N}}) = 1$ for all m and hence $\mathrm{Sp}(A(a))$ is countable. It follows that $\Xi_{\mathrm{Lm}}(A(a)) = 0$. On the other hand, if $\Xi_{3,Q}(\{a_{m,i,j}\}_{m,i,j\in\mathbb{N}}) = 0$, then there exists some $m \in \mathbb{N}$ such that $\Xi_{2,Q}(\{a_{m,i,j}\}_{i,j\in\mathbb{N}}) = 0$. In this case, $\mathrm{Sp}(A(a))$ contains an interval of length 2 so that $\Xi_{\mathrm{Lm}}(A(a)) \geq 2$. Suppose for a contradiction that $\{\Gamma_{n_3,n_2,n_1}\}$ is a Δ_4^G-tower of algorithms for $\{\Xi_{\mathrm{Lm}}, \Omega_{\mathrm{SA}}, \mathbb{R}_{\geq 0} \cup \{+\infty\}, \Lambda\}$. Given $a \in \Omega'$, each entry of $A(a)$ can be computed by evaluating finitely many of the entries of $\{a_{m,i,j}\}$. It follows that $a \mapsto 1 - \min\{\Gamma_{n_3,n_2,n_1}(A(a)), 1\}$ provides a Δ_4^G-tower of algorithms for the computational problem $\{\Xi_{3,Q}, \Omega', [0,1], \Lambda'\}$, the required contradiction.

Exercise 7.12

See the paper *"On the computation of geometric features of spectra of linear operators on Hilbert spaces,"* Foundations of Computational Mathematics, 24, 723–804, 2024.

8 Chapter 8

Exercise 8.1

Suppose for a contradiction that A is a bounded operator on $\mathcal{H}$ (infinite-dimensional) with $\mathrm{Sp}_{\mathrm{ess},1}(A) = \emptyset$. Then $\Delta_1(A) = \mathbb{C}$. By definition, we have $\Delta_5(A) = \mathbb{C}$. Hence, $\mathrm{Sp}(A) = \mathrm{Sp}_{\mathrm{d}}(A)$. For each $\lambda \in \mathrm{Sp}_{\mathrm{d}}(A)$, there exists $\epsilon_\lambda > 0$ such that $B_{\epsilon_\lambda}(\lambda) \cap B_{\epsilon_\mu}(\mu) = \emptyset$ for every distinct $\lambda, \mu \in \mathrm{Sp}_{\mathrm{d}}(A)$. It follows that $\cup_{\lambda \in \mathrm{Sp}(A)} D_{\epsilon_\lambda}(\lambda)$ cover $\mathrm{Sp}(A)$ and do not have a strict subcover. By compactness of $\mathrm{Sp}(A)$, it follows that $\mathrm{Sp}(A)$ is a finite point set. But then

$$\sum_{\lambda \in \mathrm{Sp}(A)} \mathcal{P}_\lambda = I,$$

which implies that at least one of the Riesz projections $\mathcal{P}_\lambda$ cannot have finite rank, a contradiction.

For an example of an unbounded self-adjoint operator B on $\mathcal{H}$ with empty essential spectrum, take the harmonic oscillator on $L^2(\mathbb{R})$, which is diagonalised by the basis of Hermite functions.

Let B be an unbounded self-adjoint operator on $\mathcal{H}$ with empty essential spectrum, and $z \notin \mathrm{Sp}(B)$. The above shows that $\mathrm{Sp}(A) = \mathrm{Sp}_{\mathrm{d}}(A)$ and, hence, we may write

$$B = \sum_{n=1}^{\infty} \lambda_n \mathcal{P}_{\lambda_n},$$

where $\lim_{n\to\infty} |\lambda_n| = \infty$. It follows that

$$(B - zI)^{-1} = \sum_{n=1}^{\infty} (\lambda_n - z)^{-1} \mathcal{P}_{\lambda_n}.$$

Note that (using the orthogonality of the Riesz projections)

$$\left\| (B - zI)^{-1} - \sum_{n=1}^{N} (\lambda_n - z)^{-1} \mathcal{P}_{\lambda_n} \right\| \leq \sup_{n>N} |(\lambda_n - z)^{-1}|,$$

which converges to 0 as $N \to \infty$. Hence, $(B - zI)^{-1}$ is the limit (in the operator norm topology) of a sequence of finite rank operators and so is compact.

Exercise 8.2

Let A be a normal operator on $\mathcal{H}$, $\mathcal{E}$ be the projection-valued spectral measure of A and $z \in \mathrm{Sp}(A)$.

Suppose first that $z \in \mathrm{Sp}_{\mathrm{d}}(A)$. Since z is an isolated point of the spectrum, we may choose ϵ sufficiently small so that

$$\mathcal{E}(\{w \in \mathbb{C} : |w - z| < \epsilon\}) = \mathcal{E}(\{z\}).$$

Using the functional calculus, we see that

$$\mathcal{P}_z = \frac{-1}{2\pi \mathrm{i}} \int_{|z-\lambda|=\epsilon/2} (A - \lambda I)^{-1} \,\mathrm{d}\lambda = \frac{-1}{2\pi \mathrm{i}} \int_{|z-\lambda|=\epsilon/2} \int_{\mathrm{Sp}(A)} (w - \lambda)^{-1} \,\mathrm{d}\mathcal{E}(w) \,\mathrm{d}\lambda = \int_{\mathrm{Sp}(A)} \int_{|z-\lambda|=\epsilon/2} \frac{-(w - \lambda)^{-1}}{2\pi \mathrm{i}} \,\mathrm{d}\lambda \,\mathrm{d}\mathcal{E}(w).$$

The exchange of integrals in the last equality is valid by Fubini's theorem. By Cauchy's integral formula, the inner integral is 1 if $w = z$ and it is 0 if $w \in \mathrm{Sp}(A) \setminus \{z\}$. It follows that $\mathcal{E}(\{z\}) = \mathcal{P}_z$, which has finite rank.

Now suppose that $z \in \mathrm{Sp}_{\mathrm{ess}}(A)$. If z is isolated, we may perform the above computation to see that there exists some $\epsilon > 0$ with $\mathcal{P}_z = \mathcal{E}(\{w \in \mathbb{C} : |w - z| < \epsilon\}) = \mathcal{E}(\{z\})$. Since $\mathcal{P}_z$ cannot have finite rank, $\mathcal{E}(\{w \in \mathbb{C} : |w - z| < \epsilon\})$ has infinite-dimensional range for every $\epsilon > 0$. If z is not an isolated point of $\mathrm{Sp}(A)$, we may choose a countable number of disjoint open subsets $U_j \subset \{w \in \mathbb{C} : |w - z| < \epsilon\}$ that intersect the spectrum. By the spectral theorem, $\mathcal{E}(U_j) \neq 0$. Moreover, these projections are pairwise orthogonal. Since the range of $\mathcal{E}(U_j)$ is contained in $\mathcal{E}(\{w \in \mathbb{C} : |w - z| < \epsilon\})$, we see that $\mathcal{E}(\{w \in \mathbb{C} : |w - z| < \epsilon\})$ has infinite-dimensional range for every $\epsilon > 0$.

Exercise 8.3

For the first part, note that by summing a geometric series, we have

$$\frac{1}{n}\sum_{k=1}^{n} z^{-k}A^k x = \left[\int_{\mathrm{Sp}(A)} \frac{1}{n}\sum_{k=1}^{n} z^{-k}\lambda^k \,\mathrm{d}\mathcal{E}(\lambda)\right] x = \mathcal{E}(\{z\})x + \left[\int_{\mathrm{Sp}(A)\setminus\{z\}} \frac{\lambda(1-(\lambda/z)^n)}{n(z-\lambda)} \,\mathrm{d}\mathcal{E}(\lambda)\right] x.$$

Let $f_n(\lambda) = \frac{1}{n}\frac{\lambda(1-(\lambda/z)^n)}{z-\lambda}$. Then

$$\left\|\frac{1}{n}\sum_{k=1}^{n} z^{-k}A^k x - \mathcal{E}(\{z\})x\right\|^2 = \int_{\mathrm{Sp}(A)\setminus\{z\}} |f_n(\lambda)|^2 \,\mathrm{d}\mu_x(\lambda).$$

The functions f_n are uniformly bounded and converge to zero on $\mathrm{Sp}(A) \setminus \{z\}$. Hence, the required result follows by the dominated convergence theorem.

For the non-normal example, consider the operator

$$B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{so that} \quad \frac{1}{n}\sum_{k=1}^{n} 1^{-k}B^k = \begin{pmatrix} 1 & \frac{n+1}{2} \\ 0 & 1 \end{pmatrix},$$

which clearly does not converge.

Exercise 8.4

We first prove the lemma. The first part is an easy generalisation of the argument in Chapter 1. Suppose first that a is invertible. Let $x \in \mathcal{A}$ with $\|x\| = 1$. Then

$$1 = \|x\| = \|a^{-1}ax\| \le \|a^{-1}\|\|ax\|.$$

Upon taking the infimum over x, $\sigma_{\inf}(a) \ge \|a^{-1}\|^{-1}$. Conversely, let $x_n \in \mathcal{A}$ such that $\|x_n\| = 1$ and $\|a^{-1}x_n\| \to \|a^{-1}\|$. Then

$$1 = \|x_n\| = \|aa^{-1}x_n\| \ge \sigma_{\inf}(a)\|a^{-1}x_n\|.$$

Letting $n \to \infty$, we obtain $\sigma_{\inf}(a) \le \|a^{-1}\|^{-1}$.

Now suppose that a is not invertible. Assume, for a contradiction, that $\sigma_{\inf}(a) > 0$ and $\sigma_{\inf}(a^*) > 0$. Let $b = a^*a$, then b is positive and $\mathrm{Sp}(b) \subset [0, \infty)$. Moreover,

$$\sigma_{\inf}(b) = \inf\{\|a^*ax\| : x \in \mathcal{A}, \|x\| = 1\} \ge \sigma_{\inf}(a^*)\sigma_{\inf}(a) > 0.$$

Let $\epsilon \in (0, \sigma_{\inf}(b)/2)$, then $b + \epsilon 1$ is invertible and

$$\sigma_{\inf}(b + \epsilon 1) = \inf\{\|bx + \epsilon x\| : x \in \mathcal{A}, \|x\| = 1\} \ge \sigma_{\inf}(b) - \epsilon > \epsilon.$$

From the first part of the lemma, we see that

$$\|\epsilon(b + \epsilon 1)^{-1}\| = \epsilon\|(b + \epsilon 1)^{-1}\| = \epsilon\sigma_{\inf}(b + \epsilon 1)^{-1} < 1.$$

In particular, we may define

$$c = (b + \epsilon 1)^{-1}\sum_{n=0}^{\infty} \epsilon^n (b + \epsilon 1)^{-n}.$$

An elementary bit of algebra shows that

$$cb = 1 + (cb - 1)\epsilon(b + \epsilon 1)^{-1}.$$

Iterating, we obtain

$$cb - 1 = (cb - 1)[\epsilon(b + \epsilon 1)^{-1}]^N \quad \forall N \in \mathbb{N}.$$

Taking $N \to \infty$, we see that $cb = 1$. In particular, $(ca^*)a = 1$ so that ca^* is a left inverse for a. By considering aa^*, we can argue in a similar fashion to produce a right inverse for a. It follows that a is invertible, the required contradiction.

For the next part, we first claim that if $A \in \Omega_{\mathrm{B}}$, then

$$\|\pi(A)\| = \lim_{n\to\infty} \|AQ_n^*\|.$$

To see this, let K be a compact operator on $\ell^2(\mathbb{N})$. Since $\lim_{n\to\infty} \|KQ_n^*\| = 0$,

$$\lim_{n\to\infty} \|AQ_n^*\| = \lim_{n\to\infty} \|(A+K)Q_n^*\| \le \|A+K\|.$$

Taking the infimum over K, we see that

$$\lim_{n\to\infty} \|AQ_n^*\| \le \|\pi(A)\|.$$

For the other direction, for every $n \in \mathbb{N}$, we have

$$\|AQ_n^*\| = \|A - A\mathcal{P}_n^*\| \ge \|\pi(A)\|,$$

where the first equality holds since we may write $A - A\mathcal{P}_n^*$ as a direct sum of a zero operator and AQ_n^*, and the inequality holds since $A\mathcal{P}_n^*$ is compact. Taking $n \to \infty$, we have $\lim_{n\to\infty} \|AQ_n^*\| \ge \|\pi(A)\|$. It follows that we may write $\sigma_{\inf}(\pi(A))$ as

$$\sigma_{\inf}(\pi(A)) = \inf\left\{\lim_{n\to\infty} \|ATQ_n^*\| : T \in \Omega_{\mathrm{B}}, \lim_{m\to\infty} \|TQ_m^*\| = 1\right\}.$$

Recall that

$$\tau_{\inf}(A) = \inf\left\{\liminf_{n\to\infty} \|Ax_n\| : \{x_n\} \subset \mathcal{D}(A), \|x_n\| = 1, x_n \xrightarrow{w} 0\right\}.$$

Fix $T \in \Omega_{\mathrm{B}}$ with $\lim_{m\to\infty} \|TQ_m^*\| = 1$. We may choose x_n in the range of Q_n with $\|x_n\| = 1$ and $\lim_{n\to\infty} \|Tx_n\| = 1$. For sufficiently large n, define $y_n = Tx_n/\|Tx_n\|$. Then $y_n \xrightarrow{w} 0$ and

$$\|Ay_n\| = \frac{\|ATQ_n^*Q_nx_n\|}{\|Tx_n\|} \le \frac{\|ATQ_n^*\|}{\|Tx_n\|}.$$

Taking $n \to \infty$, we see that

$$\tau_{\inf}(A) \le \liminf_{n\to\infty} \|Ay_n\| \le \lim_{n\to\infty} \|ATQ_n^*\|.$$

Taking the infimum over such T, we have

$$\tau_{\inf}(A) \le \sigma_{\inf}(\pi(A)).$$

For the other inequality, we may choose x_n with $\|x_n\| = 1$ and $x_n \xrightarrow{w} 0$, such that

$$\lim_{n\to\infty} \|Ax_n\| = \tau_{\inf}(A).$$

We may also ensure that there are disjoint index sets

$$I_n = \{a_n, a_n + 1, \dots, b_n\} \subset \mathbb{N}, \quad a_{n+1} = b_n + 1, \quad n = 1, 2, 3, \dots,$$

so that $x_n \in \operatorname{span}\{e_i : i \in I_n\}$. We may also ensure that the vectors Ax_m are asymptotically orthogonal in the sense that

$$\lim_{n\to\infty} \sum_{m=n}^{\infty} \sum_{j=n, j\neq m}^{\infty} |\langle Ax_m, Ax_j\rangle|^2 = 0.$$

Let

$$T = \sum_{m=1}^{\infty} x_m x_m^*,$$

then $T \in \Omega_{\mathrm{B}}$ with $\lim_{m\to\infty} \|TQ_m^*\| = 1$. If x is in the range of Q_{a_n-1}, then

$$\|ATQ_{a_n-1}^*x\|^2 = \sum_{m=n}^{\infty} \sum_{j=n}^{\infty} \langle Ax_m, Ax_j\rangle\langle x, x_m\rangle\langle x_j, x\rangle \le \sum_{m=n}^{\infty} \|Ax_m\|^2 |\langle x, x_m\rangle|^2 + \|x\|^2 \sum_{m=n}^{\infty} \sum_{j=n, j\neq m}^{\infty} |\langle Ax_m, Ax_j\rangle|^2.$$

It follows that

$$\|ATQ_{a_n-1}^*\| \le \sqrt{\sum_{m=n}^{\infty} \sum_{j=n, j\neq m}^{\infty} |\langle Ax_m, Ax_j\rangle|^2} + \sup_{m\ge n} \|Ax_m\|.$$

The limit supremum of the right-hand side is $\tau_{\inf}(A)$, and hence $\sigma_{\inf}(\pi(A)) \le \lim_{n\to\infty} \|ATQ_{a_n-1}^*\| \le \tau_{\inf}(A)$.

Exercise 8.5

The proof of the first two bullet points is almost verbatim the proof of the stated Lemma from Chapter 1. For the final part, let $A \in \Omega_f$. We note that

$$\|\mathcal{P}_{f(n_1)} A Q^*_{n_3,n_1} - A Q^*_{n_3,n_1}\| \leq \|\mathcal{P}_{f(n_1)} A \mathcal{P}^*_{n_1} - A \mathcal{P}^*_{n_1}\| \leq c_{n_1}.$$

It follows that

$$\tau_{n_3,n_1}(A - zI) \leq \tau_{n_3,n_1,f(n_1)}(A - zI) + c_{n_1} \leq \tau_{n_3,n_1}(A - zI) + c_{n_1}.$$

Hence, $\tau_{n_3,n_1,f(n_1)}(A - zI) + c_{n_1}$ converges (uniformly on compact subsets of $\mathbb{C}$) from above to $\tau_{n_3}(A - zI)$ as $n_1 \to \infty$.

Exercise 8.6

Let A be a normal operator, $z \in \mathbb{C}$ and $n \in \mathbb{N}$. If there are n eigenvalues of A (counted according to multiplicity) that are closer to z than $\mathrm{Sp}_{\mathrm{ess}}(A)$, then we may write

$$A - zI = (B - zI) \oplus \sum_{j=1}^{n} (\lambda_j - z) v_j v_j^*,$$

for orthonormal eigenvectors $\{v_1, \ldots, v_n\}$ and a normal operator B with $\mathrm{dist}(z, \mathrm{Sp}(B)) \geq \sup\{|\lambda_j - z| : j = 1, \ldots, n\}$. Here, $\lambda_1, \ldots, \lambda_n$ are the closest n eigenvalues to z (if there are ties then we may select these arbitrarily). It is then easy to see that

$$\xi_n(z, A) = \sup\{|\lambda_j - z| : j = 1, \ldots, n\},$$

which is the distance of z to the nearest point in the spectrum of A, after disregarding $n - 1$ points counted according to multiplicity. Suppose now that there are not n eigenvalues of A (counted according to multiplicity) that are closer to z than $\mathrm{Sp}_{\mathrm{ess}}(A)$. Let $\lambda_1, \ldots, \lambda_k$ be the eigenvalues that are closer to z than $\mathrm{Sp}_{\mathrm{ess}}(A)$. Then we may write

$$A - zI = (B - zI) \oplus \sum_{j=1}^{k} (\lambda_j - z) v_j v_j^*$$

in a similar fashion. We then see that

$$\xi_n(z, A) \geq \mathrm{dist}(z, \mathrm{Sp}(B)) = \mathrm{dist}(z, \mathrm{Sp}_{\mathrm{ess}}(A)).$$

On the other hand, Exercise 8.2 shows that for every $\epsilon > 0$, $\mathcal{E}(B_{\mathrm{dist}(z,\mathrm{Sp}_{\mathrm{ess}}(A))+\epsilon}(z))$ has infinite rank. It follows that $\xi_n(z, A) \leq \mathrm{dist}(z, \mathrm{Sp}_{\mathrm{ess}}(A)) + \epsilon$. Since $\epsilon > 0$ was arbitrary, $\xi_n(z, A) = \mathrm{dist}(z, \mathrm{Sp}_{\mathrm{ess}}(A))$. This finishes the proof.

Exercise 8.7

For the upper bounds, we consider the towers of algorithms for Ξ_{mult} (the sum of the multiplicity over the discrete spectrum). For example, for $\Omega_{\mathrm{N}} \cap \Omega_f$, there is a Σ_2^A-tower for $\{\Xi_{\mathrm{mult}}, \Omega_{\mathrm{N}} \cap \Omega_f, \mathbb{Z}_{\geq 0} \cup \{+\infty\}\}$. Call this tower $\{\Gamma'_{n_2,n_1}\}$. We then set

$$\Gamma_{n_2,n_1}(A) = \begin{cases} 1, & \text{if } \Gamma'_{n_2,n_1}(A) > 0, \\ 0, & \text{otherwise.} \end{cases}$$

This provides a Σ_2^A-tower for the decision problem, "Is Sp_{d} non-empty?" for $\Omega_{\mathrm{N}} \cap \Omega_f$. The class Ω_{N} is analogous but requires an additional limit.

For the lower bounds, we argue first for Ω_{D}. Suppose for a contradiction that $\{\Gamma_n\}$ is a Δ_2^G-tower for $\{\Xi, \Omega_{\mathrm{D}}, \mathcal{M}_{\mathrm{dec}}\}$. Consider the matrix operators $A_m = \mathrm{diag}\{0, 0, \ldots, 0, 2\} \in \mathbb{C}^{m \times m}$ and $C = \mathrm{diag}\{0, 0, \ldots\}$ and set

$$A = \bigoplus_{m=1}^{\infty} A_{k_m},$$

where we choose an increasing sequence k_m inductively as follows. Set $k_1 = 1$ and suppose that $k_1, ..., k_m$ have been chosen. $\mathrm{Sp}_d(A_{k_1} \oplus A_{k_2} \oplus \cdots \oplus A_{k_m} \oplus C) = \{2\}$ so there exists some $n_m \geq m$ such that if $n \geq n_m$ then

$$\Gamma_n(A_{k_1} \oplus \cdots \oplus A_{k_m} \oplus C) = 1.$$

Now let $k_{m+1} \geq \max\{N(A_{k_1} \oplus \cdots \oplus A_{k_m} \oplus C, n_m), k_m + 1\}$. Arguing as usual, it follows that $\Gamma_{n_m}(A) = \Gamma_{n_m}(A_{k_1} \oplus \cdots \oplus A_{k_m} \oplus C)$. But $\Gamma_{n_m}(A)$ converges to 0 as A has no discrete spectrum, a contradiction.

To prove the lower bound for Ω_{SA}, we can use a slight alteration of the argument used in the book to prove $\{\mathrm{Sp}_{\mathrm{ess}}, \Omega_{\mathrm{SA}}, \mathcal{M}_{\mathrm{AW}}\} \notin \Delta_3^G$. Replace the A there by

$$A = \mathrm{diag}\{1, 0, 2, 0, 2, \ldots\} \bigoplus \left[\bigoplus_{j=1}^{\infty} C(\{a_{i,j}\}_{i \in \mathbb{N}}) \right]$$

and argue analogously.

Exercise 8.8

Let A be a closed and densely defined operator on $\mathcal{H}$. We first prove that

$$\mathrm{Sp}_{\mathrm{ess},2}(A) = \{z \in \mathbb{C} : 0 \in \mathrm{Sp}_{\mathrm{ess}}((A - zI)^*(A - zI))\} .$$

Suppose first that $z \in \mathbb{C}$ is such that $0 \notin \mathrm{Sp}_{\mathrm{ess}}((A - zI)^*(A - zI))$. In other words, $(A - zI)^*(A - zI)$ is Fredholm. We have $\ker(A - zI) \subset \ker((A - zI)^*(A - zI))$. If $x \in \ker((A - zI)^*(A - zI))$, then

$$\|(A - zI)x\|^2 = \langle (A - zI)x, (A - zI)x \rangle = \langle (A - zI)^*(A - zI)x, x \rangle = 0.$$

Hence, $\ker((A - zI)^*(A - zI)) \subset \ker(A - zI)$ and, in fact, $\ker(A - zI) = \ker((A - zI)^*(A - zI))$. It follows that $\mathrm{nul}(A - zI) = \mathrm{nul}((A - zI)^*(A - zI)) < \infty$. The operator $(A - zI)^*(A - zI)$ is self-adjoint, positive and is bounded away from 0 on $\mathcal{D}((A - zI)^*(A - zI)) \cap \ker(A - zI)^{\perp}$. Hence, there exists $C > 0$ such that for every $x \in \mathcal{D}((A - zI)^*(A - zI)) \cap \ker(A - zI)^{\perp}$,

$$\|(A - zI)x\|^2 = \langle (A - zI)^*(A - zI)x, x \rangle \geq C\|x\|^2.$$

Since $\mathcal{D}((A - zI)^*(A - zI))$ forms a core for $A - zI$, it follows that this bound can be extended to every $x \in \mathcal{D}(A - zI) \cap \ker(A - zI)^{\perp}$. Suppose for a contradiction that the range of $A - zI$ is not closed. Then there exists some $y_n = (A - zI)x_n$ with $\lim_{n\to\infty} y_n = y$ and $x_n \in \mathcal{D}(A - zI) \cap \ker(A - zI)^{\perp}$, but $y \notin \mathrm{ran}(A - zI)$. It follows from the above bound that $\{x_n\}$ is Cauchy and hence converges. Closedness of A implies that $y \in \mathrm{ran}(A - zI)$, a contradiction. It follows that $A - zI \in \mathscr{F}_+(\mathcal{H})$ so that $z \notin \mathrm{Sp}_{\mathrm{ess},2}(A)$. In particular, we have shown that

$$\mathrm{Sp}_{\mathrm{ess},2}(A) \subset \{z \in \mathbb{C} : 0 \in \mathrm{Sp}_{\mathrm{ess}}((A - zI)^*(A - zI))\} .$$

For the reverse inclusion, suppose that $z \in \mathbb{C}$ is such that $0 \in \mathrm{Sp}_{\mathrm{ess}}((A - zI)^*(A - zI))$. Then there exists a singular sequence $\{x_n\}$ of $(A - zI)^*(A - zI)$ corresponding to 0. But then

$$\|(A - zI)x_n\|^2 = \langle (A - zI)^*(A - zI)x_n, x_n \rangle \leq \|(A - zI)^*(A - zI)x_n\| \to 0$$

as $n \to \infty$. Hence, $\{x_n\}$ is a singular sequence of A corresponding to z, which implies that $z \in \mathrm{Sp}_{\mathrm{ess},2}(A)$.

We then have

$$\begin{aligned} \mathrm{Sp}_{\mathrm{ess},1}(A) &= \{z \in \mathbb{C} : z \in \mathrm{Sp}_{\mathrm{ess},2}(A) \text{ and } \bar{z} \in \mathrm{Sp}_{\mathrm{ess},2}(A^*)\} \\ &= \{z \in \mathbb{C} : 0 \in \mathrm{Sp}_{\mathrm{ess}}((A - zI)^*(A - zI)) \cap \mathrm{Sp}_{\mathrm{ess}}((A - zI)(A - zI)^*)\} \end{aligned}$$

and

$$\begin{aligned} \mathrm{Sp}_{\mathrm{ess},3}(A) &= \{z \in \mathbb{C} : z \in \mathrm{Sp}_{\mathrm{ess},2}(A) \text{ or } \bar{z} \in \mathrm{Sp}_{\mathrm{ess},2}(A^*)\} \\ &= \{z \in \mathbb{C} : 0 \in \mathrm{Sp}_{\mathrm{ess}}((A - zI)^*(A - zI)) \cup \mathrm{Sp}_{\mathrm{ess}}((A - zI)(A - zI)^*)\} . \end{aligned}$$

This proves the required equalities for essential spectra.

Now suppose that A is normal. Then

$$\mathrm{Sp}(A) = \{z \in \mathbb{C} : 0 \in \mathrm{Sp}((A - zI)^*(A - zI))\} .$$

It follows that

$$\begin{aligned}\mathrm{Sp_d}(A) = \mathrm{Sp}(A) \setminus \mathrm{Sp_{ess}}(A) &= \{z \in \mathbb{C} : 0 \in \mathrm{Sp}((A - zI)^*(A - zI)), 0 \notin \mathrm{Sp_{ess}}((A - zI)^*(A - zI))\} \\ &= \{z \in \mathbb{C} : 0 \in \mathrm{Sp_d}((A - zI)^*(A - zI))\}.\end{aligned}$$

To see why this need not hold for non-normal operators, take the shift operator on $\ell^2(\mathbb{N})$ defined by $Ae_n = e_{n-1}$ if $n > 1$ and $Ae_1 = 0$. Then $0 \notin \mathrm{Sp_d}(A)$, but $A^*A = \mathrm{diag}(0, 1, 1, \ldots)$ has $0 \in \mathrm{Sp_d}(A^*A)$.

Now let $A \in \Omega_{\mathrm{N}}$ and list the singular values of $(A - zI)\mathcal{P}_n^*$ as $\sigma_n^{(n)}(z, A) \leq \sigma_{n-1}^{(n)}(z, A) \leq \cdots \leq \sigma_1^{(n)}(z, A)$. Let $B = (A - zI)^*(A - zI)$ and $B_n = \mathcal{P}_n(A - zI)^*(A - zI)\mathcal{P}_n^*$. Then $\{\sigma_j^{(n)}(z, A) : j = n, n - 1, \ldots, 1\}$ are the square roots of the eigenvalues of B_n. Note that B is a non-negative, self-adjoint operator. Applying Rayleigh–Ritz, for each $j \in \mathbb{N} \cup \{0\}$, $\sigma_{n-j}^{(n)}(z, A)$ is decreasing in n and converges to a limit denoted by $\sigma_{\inf+j}(A - zI)$, which corresponds to the square root of the $(j + 1)$th smallest eigenvalue of B, $\sqrt{\lambda_{j+1}(B)}$. There are three cases to consider:

(a) If $h(z, A) = 0$, then $\lambda_1(B) > 0$.

(b) If $h(z, A) = +\infty$, the above characterisation shows that $0 \in \mathrm{Sp_{ess}}(B)$ and, hence, $\lambda_j(B) = 0$ for all j.

(c) If $h(z, A) \in \mathbb{N}$, the above characterisation shows that $\lambda_j(B) = 0$ for $j = 1, \ldots, h(z, A)$, but $\lambda_j(B) > 0$ for $j > h(z, A)$.

If (a) holds, then there exists $\delta > 0$ such that $\sigma_{\inf+j}(A - zI) \geq \delta$ for all j. Hence, $\sum_{j=0}^{k} \max\{0, 1 - k \times \sigma_{\inf+j}(A - zI)\} = 0$ for $k \geq 1/\delta$. If (c) holds, we can argue similarly to see that $\sum_{j=0}^{k} \max\{0, 1 - k \times \sigma_{\inf+j}(A - zI)\} = h(z, A)$ for sufficiently large k. If (b) holds, then $\sum_{j=0}^{k} \max\{0, 1 - k \times \sigma_{\inf+j}(A - zI)\} = (k + 1)$ diverges to $h(z, A)$.

Exercise 8.9

The norm $\|B\|$ is the square root of the largest eigenvalue of the semi-positive definite self-adjoint matrix B^*B. This can be estimated from above to an accuracy of 1 (using the techniques of Section 3.3 of the book). Hence, we can compute an upper bound $M \in \mathbb{Q}_+$ for $\|B\|$ in finitely many arithmetic operations and comparisons. Now choose points $x_1, \ldots, x_k \in \mathbb{Q}^n$, each of norm at most 1, such that $d_{\mathrm{H}}(\{x_1, \ldots, x_k\}, \{x \in \mathbb{C}^n : \|x\| = 1\}) < \epsilon/(3M)$. These can be computed in finitely many arithmetic operations and comparisons using generalised polar coordinates and approximations of trigonometric identities. It follows that

$$d_{\mathrm{H}}(\{\langle Bx_1, x_1\rangle, \ldots, \langle Bx_k, x_k\rangle\}, W(B)) \leq 2\epsilon/3.$$

We then let each $z_j \in \mathbb{Q} + \mathrm{i}\mathbb{Q}$ be a $\epsilon/4$ approximation of $\langle Bx_j, x_j\rangle$, which can be computed in finitely many arithmetic operations and comparisons.

Let $A \in \Omega_{\ell^2(\mathbb{N})}$ and $\mathcal{P}_n$ be the orthogonal projection onto $\mathrm{span}\{e_1, \ldots, e_n\}$. Then $W(\mathcal{P}_nA\mathcal{P}_n^*) \uparrow \mathrm{Cl}(W(A))$ in the Attouch–Wets topology as $n \to \infty$. We then let $\Gamma_n(A)$ be an approximation of $W(\mathcal{P}_nA\mathcal{P}_n^*)$ with $d_{\mathrm{H}}(\Gamma_n(A), W(\mathcal{P}_nA\mathcal{P}_n^*)) \leq 1/n$. This provides the desired Σ_1^A-tower.

Exercise 8.10

Since $T\psi_m = (2m + 1)\psi_m + (\mathrm{i} - 1)x^2\psi_m$, we have

$$\langle Tf, f\rangle = \sum_{m=n+1}^{\infty} \sum_{k=n+1}^{\infty} c_m\overline{c_k}\langle(2m + 1)\psi_m + (\mathrm{i} - 1)x^2\psi_m, \psi_k\rangle = (\mathrm{i} - 1)\|xf\|_{L^2(\mathbb{R})}^2 + \sum_{m=n+1}^{\infty} (2m + 1)|c_m|^2,$$

where we have used the fact that the Hermite functions are orthonormal. Note that

$$\sum_{m=n+1}^{\infty} (2m + 1)|c_m|^2 \geq (2n + 3) \sum_{m=n+1}^{\infty} |c_m|^2 = 2n + 3.$$

Hence, if $|\mathrm{Re}(\langle Tf, f\rangle)| \leq n + 1$, then

$$\|xf\|_{L^2(\mathbb{R})}^2 \geq 2n + 3 - (n + 1) = n + 2.$$

This implies that $|\mathrm{Im}(\langle Tf, f\rangle)| = \|xf\|_{L^2(\mathbb{R})}^2 \geq n + 2$. It follows that $|\langle Tf, f\rangle| \geq n$. Hence, there are radii R_n with $\lim_{n\to\infty} R_n = \infty$ such that if $|z| \leq R_n$, then $z \notin W(Q_nTQ_n^*)$. We know that if $W_e(A) \neq \emptyset$, then $\mathrm{Cl}(W(Q_nAQ_n^*)) \downarrow W_e(A)$ in the Attouch–Wets topology as $n \to \infty$. Hence, we must have $W_e(T) = \emptyset$.

Exercise 8.11

Suppose that $\{\mathcal{P}_n\}$ is a sequence of orthogonal projections onto finite-dimensional subspaces such that $\mathcal{P}_n^*\mathcal{P}_n$ converges strongly to the identity, and such that $\lambda_n \in \mathrm{Sp}(\mathcal{P}_n A \mathcal{P}_n^*)$ have $\lambda_n \to \lambda \notin \mathrm{Sp}(A)$. Exercise 1.1 shows that $x_n = \mathcal{P}_n^* v_n$ converges weakly to zero as $n \to \infty$, where v_n are the normalised ($\|v_n\| = 1$) eigenvectors of $\mathcal{P}_n A \mathcal{P}_n^*$ corresponding to λ_n. Note that

$$\langle Ax_n, x_n\rangle = \langle A\mathcal{P}_n^* v_n, \mathcal{P}_n^* v_n\rangle = \langle \mathcal{P}_n A \mathcal{P}_n^* v_n, v_n\rangle = \lambda_n.$$

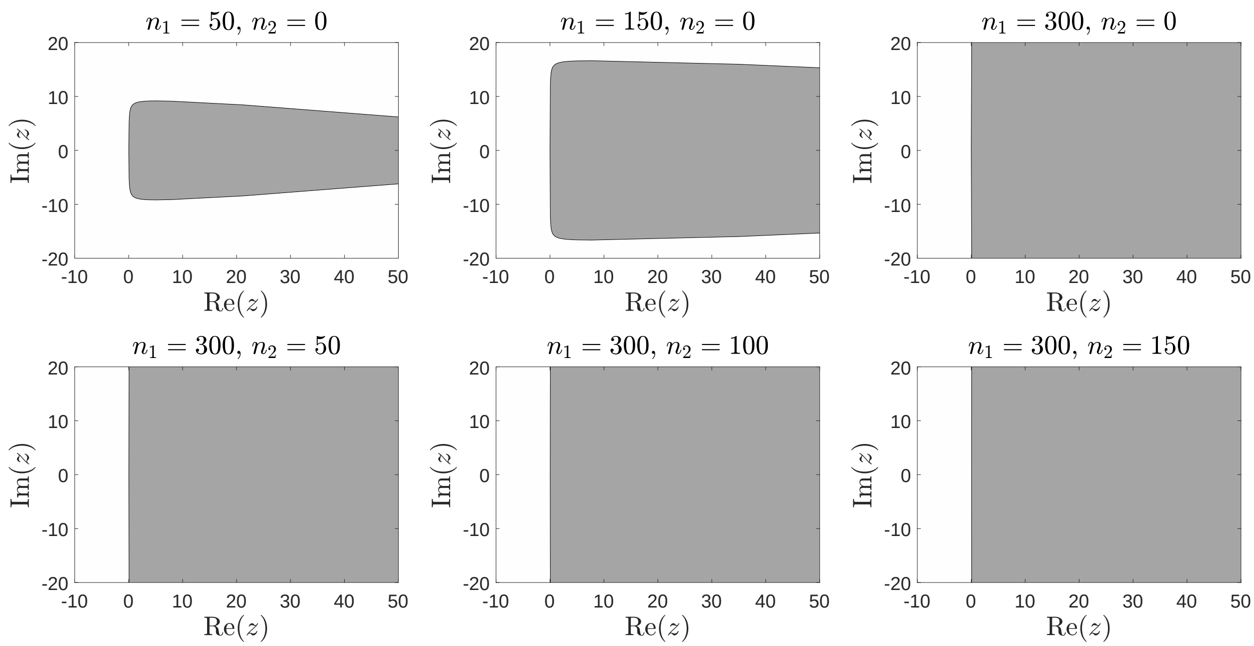
Now fix $m \in \mathbb{N}$ and set $Q_m = I - \mathcal{P}_m$. Set $y_n = Q_m^* Q_m x_n$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$ and, hence,

$$\lim_{n\to\infty} \langle Ay_n, y_n\rangle = \lim_{n\to\infty} \langle Ax_n, x_n\rangle = \lim_{n\to\infty} \lambda_n = \lambda.$$

It follows that $\lambda \in \mathrm{Cl}\left(W(Q_m A Q_m^*)\right)$. Since $m \in \mathbb{N}$ was arbitrary, $\lambda \in W_e(A)$.

Exercise 8.12

Code for this exercise can be found in "ex8_12.m" in the repository. Here are the outputs of the tower of algorithms for $W_e(T)$ for various choices of n_1 and n_2:



We show that

$$W_e(T) = \mathrm{Cl}\left(W(T)\right) = \{z \in \mathbb{C} : \mathrm{Re}(z) \geq 0\}.$$

The inclusion '$\subset$' is obvious. Since $W_e(T)$ is closed, it remains to be proved that $\{z \in \mathbb{C} : \mathrm{Re}(z) > 0\} \subset W_e(T)$. Let $z = u + \mathrm{i}v \in \mathbb{C}$ with $u = \mathrm{Re}(z) > 0$ and $v \in \mathbb{R}$. If $\varphi \in C_c^\infty(\mathbb{R})$ is an even or odd function with $\mathrm{supp}\,\varphi \subset [-1, 1]$ and $\|\varphi\|^2 = 1/2$, $\|\varphi'\|^2 = u/2$, we define $(f_n)_{n\in\mathbb{N}} \subset \mathcal{D}(T)$ by

$$f_n(x) = \begin{cases} \varphi(x - (-n)) + \varphi(x - (n + 2|v|)), & v \geq 0, \\ \varphi(x - (-n - 2|v|)) + \varphi(x - n), & v < 0, \end{cases} \qquad x \in \mathbb{R}, \quad n \in \mathbb{N}.$$

Then $\langle \mathrm{i}xf_n, f_n\rangle = \mathrm{i}v$, $\|f_n\| = 1$, f_n converges weakly to 0 as $n \to \infty$ and

$$\langle Tf_n, f_n\rangle = \|f_n'\|^2 + \langle \mathrm{i}xf_n, f_n\rangle = 2\|\varphi'\|^2 + \mathrm{i}v = u + \mathrm{i}v = z, \quad n \in \mathbb{N},$$

which implies that $z \in W_e(T)$, as required.

9 Chapter 9

Exercise 9.1

For the existence of Δ_2^G-algorithms, we use the fact that the finite section method converges for compact operators (Exercise 1.20). For example, to compute the spectral radius of compact A, we let $\Gamma_n(A)$ be the spectral radius of $\mathcal{P}_n A \mathcal{P}_n^*$ computed to accuracy $1/n$. By considering self-adjoint, diagonal, compact operators, it is straightforward to see that none of the computational problems are in Π_1^G. We prove that the spectral radius is not in Σ_1^G, and the other problems are similar. Suppose, for a contradiction, that $\{\Gamma_n\}$ is a Σ_1^G tower for the spectral radius. We will in fact also impose that our operators lie in Ω_g for g with $g_m(x) \leq (1-\delta)x$ (though the exercise did not demand this). Choose η small such that $(1+\eta+\eta^2)^2 \leq 1/(1-\delta)$ and set

$$C = C_m \oplus \text{diag}(0,0,\ldots), \quad C_m = \begin{pmatrix} 1+\eta^2 & & & & -\eta \\ & 0 & & & \\ & & \ddots & & \\ & & & 0 & \\ \eta^3 & & & & -\eta^2 \end{pmatrix} \in \mathbb{C}^{m\times m}.$$

The condition relating η and δ ensures that $C \in \Omega_g$. By assumption, there exists some n so that

$$\Gamma_n(\text{diag}(1+\eta^2,0,0,\ldots)) - 2^{-n} > (1+\sqrt{1+4\eta^2})/2 = \rho(C).$$

We may argue in the usual manner to choose m large so that $\Gamma_n(C) = \Gamma_n(\text{diag}(1+\eta^2,0,0,\ldots))$, the required contradiction.

Exercise 9.2

Let $A \in \Omega_{\mathrm{B}} \cap \Omega_{\mathrm{N}}$ and $A_n = \mathcal{P}_n A \mathcal{P}_n^*$ denote its finite section, where $\mathcal{P}_n$ denotes the orthogonal projection onto $\text{span}\{e_1,\ldots,e_n\}$. We have

$$\mathcal{P}_n^* A_n^k \mathcal{P}_n = \mathcal{P}_n^*[\mathcal{P}_n A \mathcal{P}_n^*]^k \mathcal{P}_n = [\mathcal{P}_n^* \mathcal{P}_n A \mathcal{P}_n^* \mathcal{P}_n]^k,$$

which converges in the strong operator topology to A^k as $n \to \infty$. It follows that if $x \in \ell^2(\mathbb{N})$ with $\|x\| = 1$, then

$$\|A^k x\| = \lim_{n\to\infty} \|\mathcal{P}_n^* A_n^k \mathcal{P}_n x\| \leq \liminf_{n\to\infty} \|A_n^k\|$$

Taking the supremum over all such x, we see that $\|A^k\| \leq \liminf_{n\to\infty}\|A_n^k\|$. Using the normality of A, $\|A_n^k\| \leq \|A_n\|^k \leq \|A\|^k = \|A^k\|$. It follows that $\|A_n^k\|$ converges to $\|A^k\|$ from below as $n\to\infty$.

For the final part, let A be the infinite Toeplitz matrix from the example, which is not normal. Let B be the Laurent operator with the same symbol as A. This operator is normal and has the same finite section structure as A. Hence, $\lim_{n\to\infty}\|A_n^k\| = \|B^k\|$, which also implies that $\|A^k\| \leq \|B^k\|$, since that part of the above argument did not use normality. The operators A^k and B^k are banded. A translation argument shows that if $x \in \text{span}\{e_n : n \in \mathbb{Z}\}$ with $\|x\| = 1$, then $\|B^k x\| \leq \|A^k\|$. Taking the supremum over all such x, we see that $\|B^k\| \leq \|A^k\|$. Hence, $\|B^k\| = \|A^k\|$, as required.

Exercise 9.3

If $A \in \Omega_{\mathrm{N}}$, then A is quasinilpotent if and only if $A = 0$. Deciding if this is the case is a Π_1^A problem, where we simply check the matrix entries in turn to see if they are nonzero. It is also clear that the problem does not lie in Δ_1^G for Ω_{D}. For the class $\Omega_{\mathrm{B}} \cap \Omega_f \cap \Omega_g$, there is a Σ_1^A tower of algorithms for the spectral radius, call it $\{\hat{\Gamma}_n\}$. We define

$$\Gamma_n(A) = \begin{cases} 1, & \text{if } \hat{\Gamma}_n(A) \leq 2^{-n}, \\ 0, & \text{otherwise}, \end{cases}$$

which provides a Π_1^A-tower for the decision problem. For the class $\Omega_{\mathrm{B}} \cap \Omega_g$, there is a Σ_2^A tower of algorithms for the spectral radius, call it $\{\hat{\Gamma}_{n_2,n_1}\}$, where we may assume the first limit is strictly monotonic from above and the second is strictly monotonic from below (the strictness deals with the fact that decisions based on ≤ are not continuous, but have limits under the strictly monotonic convergence of the argument). We define

$$\Gamma_{n_2,n_1}(A) = \begin{cases} 1, & \text{if } \hat{\Gamma}_{n_2,n_1}(A) \leq 2 \cdot 2^{-n_2}, \\ 0, & \text{otherwise}, \end{cases}$$

which provides a Π_2^A-tower for the decision problem. For the class $\Omega_\mathrm{B} \cap \Omega_f$, there is a Π_2^A tower of algorithms for the spectral radius, call it $\{\hat{\Gamma}_{n_2,n_1}\}$, where we may assume the first limit is strictly monotonic from below and the second is strictly monotonic from above. We define

$$\Gamma_{n_3,n_2,n_1}(A) = \begin{cases} 1, & \text{if } \hat{\Gamma}_{n_2,n_1}(A) \leq 2 \cdot 2^{-n_3}, \\ 0, & \text{otherwise}, \end{cases}$$

which provides a Π_3^A-tower for the decision problem. Similarly, we obtain a Π_4^A-tower for the class Ω_B.

Exercise 9.4

Gelfand's formula states that

$$\lim_{k\to\infty} \|A^k\|^{1/k} = \rho(A) \leq \|A^n\|^{1/n} \quad \forall n \in \mathbb{N}.$$

To turn this into an algorithm, we need to compute $\|A^k\|$. In general, the finite section method need not converge for this quantity (this can be seen by taking direct sums of the 2×2 non-spectraloid matrix in Example 9.1.10). Instead, we exploit the fact that we are dealing with $A \in \Omega_\mathrm{B} \cap \Omega_f$. From the solution of Exercise 7.3, there is a Δ_2^A tower, $\{\tilde{\Gamma}_n\}$ such that

$$\lim_{n\to\infty} \tilde{\Gamma}_n(A,k) = \|A^k\| \quad \forall A \in \Omega_\mathrm{B} \cap \Omega_f, k \in \mathbb{N}.$$

We then let $\Gamma_{n_2,n_1}(A)$ be an approximation of $[\tilde{\Gamma}_{n_1}(A,n_2)]^{1/n_2}$ from above to accuracy $1/n_1$. This is clearly a Π_2^A-tower. The same monotonicity properties can be observed in the referenced figure, where, for that particular operator, the finite section method can be used instead of $\tilde{\Gamma}_n$, as shown in Exercise 9.2, which essentially relies on the normality of the operator.

Exercise 9.5

The following example is due to Kakutani (published in C. E. Rickart, *"General Theory of Banach Algebras,"* 1974, pages 282–283), who used it to show that the spectral radius is not continuous with respect to the operator norm topology. We will show that the same example also works for the essential spectral radius. Let A be the weighted shift operator on $\ell^2(\mathbb{N})$ given by

$$Ae_m = \alpha_m e_{m+1}, \quad \alpha_m = e^{-k} \text{ where } m = 2^k(2l+1), \quad k, l = 0, 1, 2, \dots.$$

Then for every $s \in \mathbb{N}$ and $t \in \mathbb{N}$ with $t > s$,

$$[\sigma_{2^s}(A^{2^t-2^s})]^{1/(2^t-2^s)} \geq (\alpha_{2^s}\alpha_{2^s+1}\cdots\alpha_{2^t-1})^{1/(2^t-2^s)} = \prod_{j=1}^{t-1} \exp(-j2^{t-j-1}/(2^t-2^s)) \prod_{k=1}^{s-1} \exp(k2^{s-k-1}/(2^t-2^s)).$$

We clearly have

$$\lim_{t\to\infty} \prod_{k=1}^{s-1} \exp(k2^{s-k-1}/(2^t-2^s)) = 1,$$

whereas

$$\prod_{j=1}^{t-1} \exp(-j2^{t-j-1}/(2^t-2^s)) \geq \prod_{j=1}^{t-1} \exp(-j2^{t-j-1}/(2^{t-1})) \geq \exp\left(-\sum_{j=1}^{\infty} \frac{j}{2^j}\right) > 0.$$

Since this bound is independent of s, we may take $t \to \infty$ and then $s \to \infty$, applying Yamamoto's theorem, to see that $\rho_{\mathrm{ess}}(A) > 0$. We now let A_n be the shift $A_n e_m = \alpha'_m e_{m+1}$, where $\alpha'_m = \alpha_m$ if $m \neq 2^n(2l+1)$, and 0 otherwise. It is easily checked that A_n is nilpotent, so that $\rho(A_n) = 0$. However, $\|A - A_n\| = e^{-n}$.

Now suppose that A is normal and B (not necessarily normal) has $\|A-B\| \leq \epsilon$. By definition of the pseudospectrum, we have $\mathrm{Sp}(B) \subset \mathrm{Sp}_\epsilon(A)$. Hence, $\rho(B) \leq \rho_\epsilon(A) = \rho(A) + \epsilon$, where the final equality uses normality of A. If B is also normal, then we may reverse the roles of A and B to see that $\rho(A) \leq \rho(B) + \epsilon$, and, hence, $|\rho(A) - \rho(B)| \leq \epsilon$.

To argue for the essential spectral radius, we make use of Exercise 8.4. Let $\mathcal{A}$ be a C^*-algebra with identity and let $a \in \mathcal{A}$, then

$$\mathrm{Sp}_\epsilon(a) = \bigcup_{b \in \mathcal{A}, \|b\| \leq \epsilon} \mathrm{Sp}(a+b).$$

Now suppose that A, B are normal operators acting on $\mathcal{H}$. If $\lambda \in \mathrm{Sp}_{\mathrm{ess}}(A) = \mathrm{Sp}(\pi(A))$, then $\lambda \in \mathrm{Sp}_{\|\pi(A)-\pi(B)\|}(\pi(B))$. Since $\tau_{\inf}(B - \lambda I) = \sigma_{\inf}(\pi(B) - \lambda 1) = \sigma_{\inf}(\pi(B)^* - \overline{\lambda} 1)$ and $\tau_{\inf}(B - \lambda I) = \mathrm{dist}(\lambda, \mathrm{Sp}_{\mathrm{ess}}(B))$ (here we use B is normal),

$$\mathrm{dist}(\lambda, \mathrm{Sp}_{\mathrm{ess}}(B)) \leq \|\pi(A) - \pi(B)\| = \|\pi(A - B)\| \leq \|A - B\|.$$

We can argue with A and B reversed to see that $|\rho_{\mathrm{ess}}(A) - \rho_{\mathrm{ess}}(B)| \leq \|A - B\|$.

Exercise 9.6

Arguing as we did in the solution of Exercise 7.3 (to take care of the fact we do not a priori have bounds on $\|A\|$), we can adapt the procedure in Exercise 3.10 to see that there exists an arithmetic algorithm $\tilde{\Gamma}$ and unknown N (dependent on A) such that $\tilde{\Gamma}(A, k, n)$ has finitely many nonzero entries in each column and

$$\|\tilde{\Gamma}(A, k, n) - A^k \mathcal{P}_n^*\| \leq 1/n \quad \forall A \in \Omega_{\mathrm{B}} \cap \Omega_f \quad \text{if } n \geq N.$$

Using the stability of singular values under perturbations in the operator norm topology,

$$|\sigma_{n_2}(\tilde{\Gamma}(A, n_2, n_1)) - \sigma_{n_2}(A^{n_2} \mathcal{P}_{n_1}^*)| \leq 1/n_1$$

for sufficiently large n_1.

Given a finite matrix $B \in \mathbb{C}^{m \times n}$ and tolerance $\epsilon > 0$, we can compute $\sigma_1(B), \ldots, \sigma_n(B)$ to accuracy ϵ in finitely many arithmetic operations and comparisons. This follows from our usual argument of computing square roots of the eigenvalues of B. Let $\Gamma_{n_2,n_1}(A)$ be an approximation of $[\sigma_{n_2}(\tilde{\Gamma}(A, n_2, n_1))]^{1/n_2}$ to accuracy $1/n_1$. Then,

$$\lim_{n_1 \to \infty} \Gamma_{n_2,n_1}(A) = \lim_{n_1 \to \infty} [\sigma_{n_2}(A^{n_2} \mathcal{P}_{n_1}^*)]^{1/n_2} = [\sigma_{n_2}(A^{n_2})]^{1/n_2}.$$

Hence, $\{\Gamma_{n_2,n_1}\}$ is a Π_2^A-tower for $\{\rho_{\mathrm{ess}}, \Omega_{\mathrm{B}} \cap \Omega_f\}$. To compute $\rho_j(A)$, we replace $\Gamma_{n_2,n_1}(A)$ by an approximation of $[\sigma_j(\tilde{\Gamma}(A, n_2, n_1))]^{1/n_2}$ to accuracy $1/n_1$ and apply Yamamoto's theorem. It follows that the SCI of computing the spectral gap $\rho_1(A) - \rho_2(A)$ is at most 2. To see why it is not 1, consider operators of the form

$$A = \mathrm{diag}(1) \oplus \bigoplus_{r=1}^{\infty} J_{l_r}, \quad l_r \in \mathbb{N} \setminus \{1\},$$

where the J_{l_r} are Jordan blocks. Then $\rho_1(A) - \rho_2(A) = 1$ if the sequence $\{l_r\}$ is bounded, otherwise it is 0. We now argue as in the proof of $\{\mathrm{Sp}, \Omega_{\mathrm{TD}}, \mathcal{M}_{\mathrm{H}}, \Lambda\} \notin \Delta_2^G$.

Exercise 9.7

Now let $A \in \Omega_{\mathrm{B}}$ and $m \in \mathbb{N}$. Let $\mathcal{P}_n$ denote the orthogonal projection onto $\mathrm{span}\{e_1, \ldots, e_n\}$. If $n \geq m$, then

$$\sigma_m(\mathcal{P}_n A \mathcal{P}_n^*) = \sup_{\substack{U \subset \mathrm{ran}(\mathcal{P}_n) \\ \dim(U)=m}} \min_{\substack{x \in U \\ \|x\|=1}} \|\mathcal{P}_n A x\| \leq \sup_{\substack{U \subset \mathrm{ran}(\mathcal{P}_n) \\ \dim(U)=m}} \min_{\substack{x \in U \\ \|x\|=1}} \|\mathcal{P}_{n+1} A x\| \leq \sup_{\substack{U \subset \mathrm{ran}(\mathcal{P}_{n+1}) \\ \dim(U)=m}} \min_{\substack{x \in U \\ \|x\|=1}} \|\mathcal{P}_{n+1} A x\| \leq \sigma_m(\mathcal{P}_{n+1} A \mathcal{P}_{n+1}^*).$$

Hence, the sequence $\sigma_m(\mathcal{P}_n A \mathcal{P}_n^*)$ is increasing. A similar argument shows that $\sigma_m(\mathcal{P}_n A \mathcal{P}_n^*) \leq \sigma_m(A)$, and, hence, $\sigma_m(\mathcal{P}_n A \mathcal{P}_n^*)$ converges up to a limit. Now fix $N \in \mathbb{N}$. If $n \geq N$, then arguing similarly we have

$$\sigma_m(\mathcal{P}_n A \mathcal{P}_n^*) = \sup_{\substack{U \subset \mathrm{ran}(\mathcal{P}_n) \\ \dim(U)=m}} \min_{\substack{x \in U \\ \|x\|=1}} \|\mathcal{P}_n A x\| \geq \sup_{\substack{U \subset \mathrm{ran}(\mathcal{P}_N) \\ \dim(U)=m}} \min_{\substack{x \in U \\ \|x\|=1}} \|\mathcal{P}_n A x\| = \sigma_m(\mathcal{P}_n A \mathcal{P}_N^*).$$

Since $\lim_{n\to\infty} \|\mathcal{P}_n^*\mathcal{P}_n A\mathcal{P}_N^* - A\mathcal{P}_N^*\| = 0$, we see by continuity of singular values that

$$\lim_{n\to\infty} \sigma_m(\mathcal{P}_n A\mathcal{P}_n^*) \geq \sigma_m(A\mathcal{P}_N^*).$$

Let $\epsilon > 0$. There exists an m-dimensional subspace U of $\ell^2(\mathbb{N})$ such that

$$\min_{\substack{x\in U\\ \|x\|=1}} \|A^*x\| \geq \sigma_m(A^*) - \epsilon = \sigma_m(A) - \epsilon.$$

Then

$$\sigma_m(A\mathcal{P}_N^*) = \sigma_m(\mathcal{P}_N A^*) \geq \min_{\substack{x\in U\\ \|x\|=1}} \|\mathcal{P}_N A^* x\|.$$

When restricted to U, $\mathcal{P}_N A^*$ converges in the operator norm topology to A^* and, hence,

$$\lim_{N\to\infty} \sigma_m(A\mathcal{P}_N^*) \geq \min_{\substack{x\in U\\ \|x\|=1}} \|A^*x\| \geq \sigma_m(A) - \epsilon.$$

It follows that $\lim_{n\to\infty} \sigma_m(\mathcal{P}_n A\mathcal{P}_n^*) = \sigma_m(A)$.

Exercise 9.8

Suppose first that A is bounded and let $|z| > \|A\|$, then $(A - zI)^{-1}$ exists and is given by

$$(A - zI)^{-1} = -(I - A/z)^{-1}/z = \frac{-1}{z}\sum_{n=0}^{\infty}(z^{-1}A)^n.$$

We can bound the size of each term of this series to obtain

$$\|(A - zI)^{-1}\| \leq \frac{1}{|z|(1 - \|A\|/|z|)} = \frac{1}{|z| - \|A\|}.$$

This converges to zero as $|z| \to \infty$, and, hence, each $\mathrm{Sp}_\epsilon(A)$ is bounded.

Now suppose for a contradiction that A is closed and unbounded, but that $\mathrm{Sp}_\epsilon(A)$ is bounded for some $\epsilon > 0$. We would like to adapt the above Neumann series argument using the fact that A is a closed operator. The assumption that $\mathrm{Sp}_\epsilon(A)$ is bounded implies that $\mathrm{Sp}(A)$ is bounded and, hence, $(A - zI)^{-1}$ is holomorphic on the exterior of a disc enclosing $\mathrm{Sp}(A)$. Furthermore, $\|(A - zI)^{-1}\|$ is bounded at infinity and, hence,

$$(A - zI)^{-1} = A_0 + z^{-1}A_1 + z^{-2}A_2 + \cdots$$

for some bounded operators $A_0, A_1, \ldots$. This can be seen by considering $\langle (A - zI)^{-1}x, y\rangle$ for $x, y \in \mathcal{H}$ and computing coefficients of its Laurent series. We have

$$Az^{-1}(A - zI)^{-1} = z^{-1}(I + z(A - zI)^{-1}) = z^{-1}I + A_0 + z^{-1}A_1 + z^{-2}A_2 + \cdots.$$

Since $\lim_{|z|\to\infty} z^{-1}(A - zI)^{-1} = 0$ and $\lim_{|z|\to\infty} Az^{-1}(A - zI)^{-1} = A_0$, closedness of A implies that $A_0 = 0$. It follows that

$$(A - zI)^{-1} = z^{-1}A_1 + z^{-2}A_2 + \cdots, \quad A(A - zI)^{-1} = I + A_1 + z^{-1}A_2 + \cdots.$$

Since $\lim_{|z|\to\infty}(A - zI)^{-1} = 0$ and $\lim_{|z|\to\infty} A(A - zI)^{-1} = I + A_1$, closedness of A implies that $A_1 = -I$. It follows that

$$z(A - zI)^{-1} = -I + z^{-1}A_2 + \cdots, \quad Az(A - zI)^{-1} = A_2 + z^{-1}A_3 + \cdots.$$

Since $\lim_{|z|\to\infty} z(A - zI)^{-1} = -I$ and $\lim_{|z|\to\infty} Az(A - zI)^{-1} = A_2$, closedness of A implies that $A_2 = -A$. But then A is bounded, the required contradiction. This argument also shows that $\mathrm{Sp}_\epsilon(A)$ must be non-empty for every $\epsilon > 0$.

Exercise 9.9

Let A be a bounded operator and $c \geq \rho(A)$. We first prove that

$$\sup_{k\geq 0} c^{-k}\|A^k\| \geq \sup_{\epsilon>0} \frac{\rho_\epsilon(A)-c}{\epsilon}.$$

Let $\epsilon > 0$. If $c \geq \rho_\epsilon(A)-\epsilon$, then there is nothing to prove, so suppose that $c < \rho_\epsilon(A)-\epsilon$. Let $z \in \mathrm{Sp}_\epsilon(A)$ with $|z| = \rho_\epsilon(A)$ (which implies that $\|(A-zI)^{-1}\| = 1/\epsilon$). Then

$$-z(A-zI)^{-1} = I + \sum_{n=1}^{\infty}(z^{-1}A)^n = I + \sum_{n=1}^{\infty}\left(\frac{c}{z}\right)^n c^{-n}A^n,$$

where we have used the fact that $\lim_{n\to\infty}\|z^{-n}A^n\| = 0$ for the convergence of the series. It follows that

$$\frac{\rho_\epsilon(A)}{\epsilon} \leq 1 + \left(\sup_{k\geq 0} c^{-k}\|A^k\|\right)\sum_{n=1}^{\infty}\left|\frac{c}{z}\right|^n = 1 + \frac{\sup_{k\geq 0} c^{-k}\|A^k\|}{\rho_\epsilon(A)/c-1}.$$

Rearranging this inequality, we have

$$\sup_{k\geq 0} c^{-k}\|A^k\| \geq \frac{\rho_\epsilon(A)-c}{\epsilon}\frac{\rho_\epsilon(A)-\epsilon}{c} \geq \frac{\rho_\epsilon(A)-c}{\epsilon}.$$

Taking the supremum over $\epsilon > 0$ proves the desired result.

If $|z| > c$, let $\|(A-zI)^{-1}\| = \epsilon_0^{-1}$ then

$$(|z|-c)\|(A-zI)^{-1}\| \leq (\rho_{\epsilon_0}(A)-c)\epsilon_0^{-1} \leq \sup_{\epsilon>0}\frac{\rho_\epsilon(A)-c}{\epsilon}.$$

On the other hand

$$\sup_{\epsilon>0}\frac{\rho_\epsilon(A)-c}{\epsilon} = \sup_{\epsilon>0}\sup_{z\in\mathrm{Sp}_\epsilon(A)}\frac{|z|-c}{\epsilon} \leq \sup_{\epsilon>0}\sup_{z\in\mathrm{Sp}_\epsilon(A)}(|z|-c)\|(A-zI)^{-1}\| = \sup_{|z|>c}(|z|-c)\|(A-zI)^{-1}\|.$$

It follows that

$$\sup_{\epsilon>0}\frac{\rho_\epsilon(A)-c}{\epsilon} = \sup_{|z|>c}(|z|-c)\|(A-zI)^{-1}\|.$$

Now let A generate a strongly continuous semigroup and $c \geq \omega_0(A)$. We prove that

$$\sup_{t\geq 0} e^{-ct}\|e^{tA}\| \geq \sup_{\epsilon>0}\frac{\alpha_\epsilon(A)-c}{\epsilon} \quad \forall c \geq \omega_0(A).$$

Let $\epsilon > 0$. We assume that $c < \alpha_\epsilon(A)$, otherwise there is nothing to prove. Let z have $\mathrm{Re}(z) > \alpha_\epsilon(A) \geq \omega_0(A)$. Then

$$(zI-A)^{-1} = \int_0^\infty e^{-zt}e^{tA}\,\mathrm{d}t = \int_0^\infty e^{-ct}e^{tA}e^{(c-z)t}\,\mathrm{d}t$$

and, hence,

$$\|(zI-A)^{-1}\| \leq \int_0^\infty e^{(c-\mathrm{Re}(z))t}\,\mathrm{d}t \cdot \sup_{t\geq 0} e^{-ct}\|e^{tA}\| = \frac{\sup_{t\geq 0} e^{-ct}\|e^{tA}\|}{\mathrm{Re}(z)-c}.$$

It follows that

$$\sup_{t\geq 0} e^{-ct}\|e^{tA}\| \geq \frac{\mathrm{Re}(z)-c}{\|(zI-A)^{-1}\|^{-1}}.$$

We now take a sequence of such z, call them z_n, such that $\lim_{n\to\infty}\mathrm{Re}(z_n) = \alpha_\epsilon(A)$ and $\lim_{n\to\infty}\|(z_nI-A)^{-1}\|^{-1} = \epsilon$ to finish the proof of the inequality.

If $\mathrm{Re}(z) > c$, let $\|(A-zI)^{-1}\| = \epsilon^{-1}$ then

$$(\mathrm{Re}(z)-c)\|(A-zI)^{-1}\| \leq (\alpha_\epsilon(A)-c)\epsilon^{-1} \leq \sup_{\epsilon>0}\frac{\alpha_\epsilon(A)-c}{\epsilon} \quad \forall c \geq \omega_0(A).$$

On the other hand

$$\sup_{\epsilon>0}\frac{\alpha_\epsilon(A)-c}{\epsilon}=\sup_{\epsilon>0}\sup_{z\in\mathrm{Sp}_\epsilon(A)}\frac{\mathrm{Re}(z)-c}{\epsilon}\le\sup_{\epsilon>0}\sup_{z\in\mathrm{Sp}_\epsilon(A)}(\mathrm{Re}(z)-c)\|(A-zI)^{-1}\|=\sup_{\mathrm{Re}(z)>c}(\mathrm{Re}(z)-c)\|(A-zI)^{-1}\|.$$

It follows that

$$\sup_{\epsilon>0}\frac{\alpha_\epsilon(A)-c}{\epsilon}=\sup_{\mathrm{Re}(z)>c}(\mathrm{Re}(z)-c)\|(A-zI)^{-1}\|.$$

Exercise 9.10

Let $A\in\Omega_{\mathrm{SA}}$ and list the eigenvalues of A below the essential spectrum as $\lambda_1(A)\le\lambda_2(A)\le\cdots$ including multiplicity. If there are $N\in\{0\}\cup\mathbb{N}$ such eigenvalues, let $\lambda_{N+j}(A)=\inf\{z:z\in\mathrm{Sp}_{\mathrm{ess}}(A)\}$ for all $j\in\mathbb{N}$. Let $\mathcal{P}_n$ be the projection onto $\mathrm{span}\{e_1,\dots,e_n\}$ and $\lambda_1^{(n)}(A)\le\lambda_2^{(n)}(A)\le\cdots\le\lambda_n^{(n)}(A)$ be the eigenvalues of $\mathcal{P}_nA\mathcal{P}_n^*$ (listed including multiplicity). From the min-max theorem,

$$\lambda_j^{(n)}(A)=\inf_{\substack{V\subset\mathrm{ran}(\mathcal{P}_n)\\ \dim(V)=j}}\sup_{\substack{v\in V\\ \|v\|=1}}\langle\mathcal{P}_nA\mathcal{P}_n^*v,v\rangle=\inf_{\substack{V\subset\mathrm{ran}(\mathcal{P}_n^*\mathcal{P}_n)\\ \dim(V)=j}}\sup_{\substack{v\in V\\ \|v\|=1}}\langle Av,v\rangle\ge\inf_{\substack{V\subset\mathcal{D}(A)\\ \dim(V)=j}}\sup_{\substack{v\in V\\ \|v\|=1}}\langle Av,v\rangle=\lambda_j(A),$$

where the inequality follows from the fact that infimum is taken over a smaller set. A similar argument also shows that $\lambda_j^{(n)}(A)$ is non-increasing in n.

For the other direction, let $\epsilon>0$. There exists $V\subset\mathcal{D}(A)$ with $\dim(V)=j$ such that

$$\lambda_j(A)\ge\sup_{\substack{v\in V\\ \|v\|=1}}\langle Av,v\rangle-\epsilon.$$

We may pick a basis $\{v_1,\dots,v_j\}$ of V. Since the span of the canonical basis functions forms a core of A, there exists $v_k^{(n)}\in\mathrm{ran}(\mathcal{P}_n^*\mathcal{P}_n)$ such that

$$\lim_{n\to\infty}v_k^{(n)}=v_k,\quad\lim_{n\to\infty}Av_k^{(n)}=Av_k,\quad k=1,\dots,j.$$

Let $V^{(n)}=\mathrm{span}\{v_1^{(n)},\dots,v_j^{(n)}\}$, then $\dim(V^{(n)})=j$ for sufficiently large n. It follows that

$$\limsup_{n\to\infty}\lambda_j^{(n)}(A)\le\limsup_{n\to\infty}\sup_{\substack{v\in V^{(n)}\\ \|v\|=1}}\langle Av,v\rangle.$$

Now consider the $j\times j$ matrices defined by

$$\alpha_{ab}^{(n)}=\langle Av_b^{(n)},v_a^{(n)}\rangle,\quad\alpha_{ab}=\langle Av_b,v_a\rangle,\quad\beta_{ab}^{(n)}=\langle v_b^{(n)},v_a^{(n)}\rangle,\quad\beta_{ab}=\langle v_b,v_a\rangle,\quad a,b=1,\dots,j.$$

Then $\sup_{\substack{v\in V^{(n)}\\ \|v\|=1}}\langle Av,v\rangle$ is the maximum eigenvalue of $(\beta^{(n)})^{-1}\alpha^{(n)}$, which converges to the maximum eigenvalue of $(\beta)^{-1}\alpha$ as $n\to\infty$, which is $\sup_{\substack{v\in V\\ \|v\|=1}}\langle Av,v\rangle$. It follows that

$$\limsup_{n\to\infty}\lambda_j^{(n)}(A)\le\sup_{\substack{v\in V\\ \|v\|=1}}\langle Av,v\rangle\le\lambda_j(A)+\epsilon.$$

Since $\epsilon>0$ was arbitrary, we must have that $\lambda_j^{(n)}(A)\downarrow\lambda_j(A)$ as $n\to\infty$.

Exercise 9.11

Let $\Xi_{\mathrm{quasicompact}}^{\mathrm{dec}}(A)$ be 1 if A is quasicompact, and 0 otherwise. The proofs of the lower bounds $\{\rho_{\mathrm{ess}},\Omega_{\mathrm{D}}\}\notin\Delta_2^G$ and $\{\rho_{\mathrm{ess}},\Omega_{\mathrm{B}}\cap\Omega_g\}\notin\Delta_3^G$ carry over straightforwardly to show that $\{\Xi_{\mathrm{quasicompact}}^{\mathrm{dec}},\Omega_{\mathrm{D}}\}\notin\Delta_2^G$, $\{\Xi_{\mathrm{quasicompact}}^{\mathrm{dec}},\Omega_{\mathrm{B}}\cap\Omega_{\mathrm{N}}\}\notin\Delta_2^G$, $\{\Xi_{\mathrm{quasicompact}}^{\mathrm{dec}},\Omega_{\mathrm{B}}\cap\Omega_f\}\notin\Delta_2^G$, and $\{\Xi_{\mathrm{quasicompact}}^{\mathrm{dec}},\Omega_{\mathrm{B}}\cap\Omega_g\}\notin\Delta_3^G$. Next, we show that $\{\Xi_{\mathrm{quasicompact}}^{\mathrm{dec}},\Omega_{\mathrm{B}}\cap\Omega_{\mathrm{N}}\}\in\Sigma_2^A$. There exists a Σ_2^A-tower, $\hat{\Gamma}_{n_2,n_1}$ for $\{\Xi_{\mathrm{gap}},\Omega_{\mathrm{B}}\cap\Omega_{\mathrm{N}}\}$. We now set

$$\tilde{\Gamma}_{n_2,n_1}(A)=\min\left\{n_2\cdot\hat{\Gamma}_{n_2,n_1}(A),1\right\},$$

which provides a Σ_2^A-tower for $\{\Xi_{\mathrm{quasicompact}}^{\mathrm{dec}},\Omega_{\mathrm{B}}\cap\Omega_{\mathrm{N}},[0,1]\}$. We can alter this tower (e.g., see Exercise 2.10) to achieve convergence in $\mathcal{M}_{\mathrm{dec}}$. Arguing in a similar manner, $\{\Xi_{\mathrm{quasicompact}}^{\mathrm{dec}},\Omega_{\mathrm{D}}\}\in\Sigma_2^A$, $\{\Xi_{\mathrm{quasicompact}}^{\mathrm{dec}},\Omega_{\mathrm{B}}\cap\Omega_f\}\in\Sigma_2^A$, and $\{\Xi_{\mathrm{quasicompact}}^{\mathrm{dec}},\Omega_{\mathrm{B}}\cap\Omega_g\}\in\Sigma_3^A$.

Exercise 9.12

For the first part, we have

$$\rho((A-zI)^{-1}) = \sup_{w\in \mathrm{Sp}((A-zI)^{-1})} |w| = \sup_{\lambda\in\widetilde{\mathrm{Sp}}(A)} \frac{1}{|\lambda - z|} = \frac{1}{\mathrm{dist}(z,\widetilde{\mathrm{Sp}}(A))} = \frac{1}{\mathrm{dist}(z,\mathrm{Sp}(A))},$$

where the final equality holds since $\mathrm{Sp}(A)\neq\emptyset$.

For the part about the (n,ϵ)-pseudospectrum, suppose first that $z\notin \mathrm{Sp}(A)$, and let $B=(A-zI)$ and $n\in\mathbb{Z}_{\geq 0}$. Then

$$\beta_{n+1}(z,A) = \|B^{-2^{n+1}}\|^{-1/2^{n+1}} = \|(B^{-2^n})^2\|^{-1/2^{n+1}} \geq \|B^{-2^n}\|^{-2/2^{n+1}} = \|B^{-2^n}\|^{-1/2^n} = \beta_n(z,A).$$

Using Gelfand's formula and the first part of the exercise,

$$\beta_n(z,A) = \|B^{-2^n}\|^{-1/2^n} = \frac{1}{\|(B^{-1})^{2^n}\|^{1/2^n}} \leq \frac{1}{\rho(B^{-1})} = \mathrm{dist}(z,\mathrm{Sp}(A)), \quad \lim_{n\to\infty}\beta_n(z,A) = \mathrm{dist}(z,\mathrm{Sp}(A)).$$

If $z\in\mathrm{Sp}(A)$ instead, then $\beta_n(z,A)=\mathrm{dist}(z,\mathrm{Sp}(A))$ for all $n\in\mathbb{N}$. Dini's theorem implies that $\beta_n(z,A)$ converges to $\mathrm{dist}(z,\mathrm{Sp}(A))$ uniformly on compact subsets of $\mathbb{C}$. These results show that for every $n\in\mathbb{Z}_{\geq 0}$ and $\epsilon>0$,

$$\mathrm{Sp}_{n+1,\epsilon}(A)\subset \mathrm{Sp}_{n,\epsilon}(A), \quad \{z\in\mathbb{C}:\mathrm{dist}(z,\mathrm{Sp}(A))\leq\epsilon\}\subset \mathrm{Sp}_{n,\epsilon}(A).$$

Suppose, for a contradiction, that $\mathrm{Sp}_{n,\epsilon}(A)$ does not converge to $\{z\in\mathbb{C}:\mathrm{dist}(z,\mathrm{Sp}(A))\leq\epsilon\}$ in the Attouch–Wets topology. Then without loss of generality there exists $z_n\in\mathrm{Sp}_{n,\epsilon}(A)$ and $\delta>0$ with $\lim_{n\to\infty} z_n = z$ and $\mathrm{dist}(z,\mathrm{Sp}(A))\geq \epsilon+\delta$. However, the local uniform convergence of β_n implies that

$$\mathrm{dist}(z,\mathrm{Sp}(A)) = \lim_{n\to\infty}\beta_n(z_n,A)\leq\epsilon,$$

the required contradiction.

For the part about computations, let $A\in\Omega_{\mathrm{B}}\cap\Omega_f$. We can write

$$\beta_n(z,A) = \left[\min\left\{\sigma_{\inf}\left((A-zI)^{2^n}\right), \sigma_{\inf}\left((A^*-\bar{z}I)^{2^n}\right)\right\}\right]^{\frac{1}{2^n}}.$$

Let $m\in\mathbb{N}$ and define

$$\beta_{n,m}(z,A) = \left[\min\left\{\sigma_{\inf}\left((A-zI)^{2^n}\mathcal{P}_m^*\right), \sigma_{\inf}\left((A^*-\bar{z}I)^{2^n}\mathcal{P}_m^*\right)\right\}\right]^{\frac{1}{2^n}},$$

where, as usual, $\mathcal{P}_m$ denotes the orthogonal projection onto $\mathrm{span}\{e_1,\ldots,e_m\}$. We have $\beta_{n,m}(z,A)\geq\beta_n(z,A)$ and $\lim_{m\to\infty}\beta_{n,m}(z,A)=\beta_n(z,A)$, where the convergence is uniform on compact subsets of $\mathbb{C}$. From Exercise 3.10 (using the fact that $A\in\Omega_{\mathrm{B}}\cap\Omega_f$ and we can bound $\|A\|$), we can compute, for each $\delta>0$, sparse rectangular matrices $B_{n,m}(z)$ and $C_{n,m}(z)$ with an infinite number of rows and m columns such that

$$\left\|B_{n,m}(z)-(A-zI)^{2^n}\mathcal{P}_m^*\right\|\leq\delta, \quad \left\|C_{n,m}(z)-(A^*-\bar{z}I)^{2^n}\mathcal{P}_m^*\right\|\leq\delta.$$

It follows that (using the sparsity) we may compute approximations of $\beta_{n,m}(z,A)$ from above to accuracy $1/m$. We now invoke the convergence proof of `PseudoSpec` from Chapter 3 to see that $\{\mathrm{Sp}_{n,\epsilon},\Omega_{\mathrm{B}}\cap\Omega_f,\mathcal{M}_{\mathrm{AW}},\Lambda\}\in\Sigma_1^A$. For the class Ω_{B}, we use an extra limit to compute $\beta_{n,m}(z,A)$ by truncating rows, and, hence, obtain a Σ_2^A classification. This is sharp from the $n=0$ case and the proof can be adapted for larger n.

10 Chapter 10

Exercise 10.1

It is enough to show that

$$\left\{z \in U : \left\|T(z)^{-1}\right\|^{-1} < \epsilon\right\} = \bigcup_{E \in C_U(\epsilon)} \mathrm{Sp}(T + E).$$

Suppose first that $z \notin \mathrm{Sp}(T)$ and $\|T(z)^{-1}\|^{-1} < \epsilon$. Then there exists a vector $v \in \mathcal{H}_2$ of unit norm with $\|T(z)^{-1}v\|_{\mathcal{H}_1} > \epsilon^{-1}$. Let $u = T(z)^{-1}v \in \mathcal{H}_1$ and define the operator $E(z) = E : \mathcal{H}_1 \to \mathcal{H}_2$ by $Ex = -v\langle x, u\rangle_{\mathcal{H}_1}/\|u\|^2_{\mathcal{H}_1}$. Then, $\|E\| = 1/\|u\|_{\mathcal{H}_1} < \epsilon$ and $[T(z) + E]u = 0$ so $z \in \mathrm{Sp}(T + E)$. Notice that this perturbation is rank-one and independent of z.

For the reverse set inclusion, suppose for a contradiction that $z \in \mathrm{Sp}(T + E)$ for some $E \in C_U(\epsilon)$ but that $\|T(z)^{-1}\|^{-1} \geq \epsilon$. Then

$$\|T(z)^{-1}E(z)\| \leq \|T(z)^{-1}\|\|E(z)\| \leq \|E(z)\|/\epsilon < 1.$$

Note that $T(z) + E(z) = T(z)(I + T(z)^{-1}E(z))$. Using a Neumann series, we have

$$(I + T(z)^{-1}E(z))^{-1} = \sum_{j=0}^{\infty}(-1)^j[T(z)^{-1}E(z)]^j,$$

which converges because $\|T(z)^{-1}E(z)\| < 1$. Hence, since $T(z)$ is invertible, so too is the product $T(z)(I + T(z)^{-1}E(z))$. It follows that $z \notin \mathrm{Sp}(T + E)$, which is a contradiction.

Exercise 10.2

The proof is a generalisation of the previous exercise. Suppose first that $z_0 \notin \mathrm{Sp}(T)$ and $\|T(z_0)^{-1}\|^{-1} < \epsilon f(z_0)$. Then there exists a vector $v \in \mathcal{H}_2$ of unit norm with $\|T(z_0)^{-1}v\|_{\mathcal{H}_1} > (\epsilon f(z_0))^{-1}$. Let $u = T(z_0)^{-1}v \in \mathcal{H}_1$ and define the operator $E_0 : \mathcal{H}_1 \to \mathcal{H}_2$ by $E_0 x = -v\langle x, u\rangle_{\mathcal{H}_1}/\|u\|^2_{\mathcal{H}_1}$. Then, $\|E_0\| = 1/\|u\|_{\mathcal{H}_1} < \epsilon f(z_0)$. We now define

$$A_j = \frac{\alpha_j}{f(z_0)} E_0 \begin{cases} \overline{f_j(z_0)}/|f_j(z_0)|, & f_j(z_0) \neq 0, \\ 0, & \text{otherwise}, \end{cases} \qquad E(z) = \sum_{j=0}^{\infty} f_j(z) A_j.$$

Note that

$$\|A_j\|/\alpha_j \leq \|E_0\|/f(z_0) < \epsilon, \qquad \sum_{j=0}^{\infty} f_j(z_0) A_j = \frac{\sum_{j=0}^{\infty} \alpha_j |f_j(z_0)|}{f(z_0)} E_0 = E_0.$$

Since $[T(z_0) + E_0]u = 0$, it follows that $z_0 \in \mathrm{Sp}(T + E)$. Note that we can rewrite the perturbation as

$$E(z) = E_0 \sum_{f_j(z_0) \neq 0} \alpha_j \frac{f_j(z)\overline{f_j(z_0)}}{f(z_0)|f_j(z_0)|}.$$

This is a rank-one operator multiplied by a scalar-valued holomorphic function.

For the reverse set inclusion, suppose for a contradiction that $z \in \mathrm{Sp}(T + E)$ for some $E \in \mathcal{S}_U(\epsilon)$ but that $\|T(z)^{-1}\|^{-1} \geq \epsilon f(z)$. Since $\|A_j\| < \alpha_j \epsilon$ and $f(z) \neq 0$, it follows that

$$\|T(z)^{-1}E(z)\| < \|T(z)^{-1}\| \left(\sum_{j=0}^{\infty} \alpha_j |f_j(z)|\right) \epsilon = \|T(z)^{-1}\| f(z) \epsilon \leq 1.$$

We now use a Neumann series argument as before.

To compute the closure of this set, we simply replace each appearance of ϵ in the algorithm for pseudospectra with $\epsilon f(z)$. The proof of convergence carries over (this is immediate from the observation that we can consider $\|T(z)^{-1}\|^{-1}/f(z)$ instead of $\|T(z)^{-1}\|^{-1}$ in the proof).

Exercise 10.3

Let $\epsilon > 0$. Then there exists $x \in \mathcal{D}(A)$ with $\|x\| = 1$ and $\|Ax\| \leq \sigma_{\inf}(A) + \epsilon$. Let $\phi = (x, Ax)/\sqrt{1 + \|Ax\|^2} \in \mathrm{gr}(A)$, then there exists $\psi = (y, By) \in \mathrm{gr}(B)$ with $\|\phi - \psi\| \leq d_{\mathfrak{G}}(A, B) + \epsilon$. Hence,

$$\|By\| \leq \|Ax\|/\sqrt{1 + \|Ax\|^2} + d_{\mathfrak{G}}(A, B) + \epsilon, \quad \|y\| \geq \|x\|/\sqrt{1 + \|Ax\|^2} - d_{\mathfrak{G}}(A, B) - \epsilon.$$

It follows that

$$\sigma_{\inf}(B) \leq \frac{\|Ax\|/\sqrt{1 + \|Ax\|^2} + d_{\mathfrak{G}}(A, B) + \epsilon}{1/\sqrt{1 + \|Ax\|^2} - d_{\mathfrak{G}}(A, B) - \epsilon} \leq \frac{\sigma_{\inf}(A)/\sqrt{1 + [\sigma_{\inf}(A)]^2} + d_{\mathfrak{G}}(A, B) + 2\epsilon}{1/\sqrt{1 + [\sigma_{\inf}(A) + \epsilon]^2} - d_{\mathfrak{G}}(A, B) - \epsilon}.$$

Taking $\epsilon \downarrow 0$, we obtain

$$\sigma_{\inf}(B) \leq \frac{\sigma_{\inf}(A) + d_{\mathfrak{G}}(A, B)\sqrt{1 + [\sigma_{\inf}(A)]^2}}{1 - d_{\mathfrak{G}}(A, B)\sqrt{1 + [\sigma_{\inf}(A)]^2}}.$$

Now suppose that $\lim_{n\to\infty} d_{\mathfrak{G}}(A_n, A) = 0$. Taking $B = A_n$, the above bound shows that $\limsup_{n\to\infty} \sigma_{\inf}(A_n) \leq \sigma_{\inf}(A)$. It follows that $d_{\mathfrak{G}}(A_n, A)\sqrt{1 + [\sigma_{\inf}(A_n)]^2} < 1$ for sufficiently large n, and, hence, we may reverse the roles of A and A_n to see that

$$\sigma_{\inf}(A) \leq \frac{\sigma_{\inf}(A_n) + d_{\mathfrak{G}}(A_n, A)\sqrt{1 + [\sigma_{\inf}(A_n)]^2}}{1 - d_{\mathfrak{G}}(A_n, A)\sqrt{1 + [\sigma_{\inf}(A_n)]^2}}.$$

It follows that $\liminf_{n\to\infty} \sigma_{\inf}(A_n) \geq \sigma_{\inf}(A)$.

Exercise 10.4

Let ϕ be a smooth, non-negative, compactly supported function on $\mathbb{R}$ taking maximum value 1. Set $T(z)$ as multiplication by $1 - \phi(\cdot - 1/|z|)$ and $T(0) = I$. Let $\{z_n\} \subset \mathbb{C}$ with $\lim_{n\to\infty} z_n = z \in \mathbb{C}$. Then $T(z_n)$ converges strongly to $T(z)$ as $n \to \infty$ by the dominated convergence theorem. If we take $z = 0$ and $z_n \neq 0$, $\sigma_{\inf}(T(z_n)) = 0$ but $\sigma_{\inf}(T(z)) = 1$.

Exercise 10.5

Suppose for a contradiction that $\{\mathrm{Sp}, \Omega_{\mathrm{NL}}^{U,H}, \mathcal{M}_{\mathrm{AW}}, \Lambda_2\} \in \Delta_2^G$ with a Δ_2^G-tower of algorithms $\{\Gamma_n\}$. Consider the class Ω_{D} of bounded diagonal self-adjoint operators on $\ell^2(\mathbb{N})$. For each $A \in \Omega_{\mathrm{D}}$, we may define a constant pencil $T_A \in \Omega_{\mathrm{NL}}^{U,H}$ by setting $\langle T_A(z)e_j, \hat{e}_i\rangle_{\mathcal{H}_2} = A_{ij}$, where A_{ij} is the (i, j)th matrix entry of A with respect to the canonical basis of $\ell^2(\mathbb{N})$. Moreover, for this class of diagonal operators, Λ_2 is equivalent to the evaluation of each matrix entry A_{ij}. With this setup, it is easy to see that the decision problem of deciding whether A is invertible does not lie in Δ_2^G. We have $\mathrm{Sp}(T_A) = \emptyset$ if A is invertible and $\mathrm{Sp}(T_A) = U$ otherwise. We then set $\tilde{\Gamma}_n(A) = 1$ if $\Gamma_n(T_A) \cap D_1(x_0) \neq \emptyset$ and $\tilde{\Gamma}_n(A) = 0$ otherwise. Then $\{\tilde{\Gamma}_n\}$ is a Δ_2^G-tower for deciding if elements of Ω_{D} are invertible, a contradiction.

To deal with the evaluation set Λ_1, we embed a decision problem. Let Ω_{SA} denote the class of (linear) self-adjoint operators on $\ell^2(\mathbb{N})$ for which $\mathrm{span}\{e_n : n \in \mathbb{N}\}$ forms a core. Consider the following decision problem:

$$\Xi_0 : A \to \text{“Is } 0 \in \mathrm{Sp}(A)\text{?”} \quad A \in \Omega_{\mathrm{SA}}.$$

The following proposition classifies the difficulty of this problem.

Proposition 10.1. $\{\Xi_0, \Omega_{\mathrm{SA}}, [0, 1], \Lambda_1\} \notin \Delta_3^G$.

Proof. Let Ω' denote the collection of all infinite matrices $a = \{a_{i,j}\}_{i,j\in\mathbb{N}}$ with entries $a_{i,j} \in \{0, 1\}$. Define the problem function

$$\Xi_{2,Q}(\{a_{i,j}\}) = \begin{cases} 1, & \text{if } \sum_i a_{i,j} = \infty \text{ for all but finitely many } j, \\ 0, & \text{otherwise.} \end{cases}$$

Recall from Chapter 2 that $\{\Xi_{2,Q}, \Omega', [0, 1], \Lambda'\} \notin \Delta_3^G$, where Λ' is the set of component-wise evaluations of $\{a_{i,j}\}$.

Suppose for a contradiction that $\{\Gamma_{n_2,n_1}\}$ is a Δ_3^G-tower of algorithms for $\{\Xi_0, \Omega_{\mathrm{SA}}, [0,1], \Lambda_1\}$. Given a sequence $\{c_i\}_{i\in\mathbb{N}}$ with each $c_i \in \{0,1\}$, list the indices where $c_i = 1$ as

$$\{i : c_i = 1\} = \{i_1, i_2, \ldots\}, \quad i_1 < i_2 < \cdots .$$

This set may be empty, finite, or countably infinite. Define $i_0 = 0$ and let $r(\{c_i\})$ be one greater than the cardinality of $\{i : c_i = 1\}$. If $r(\{c_i\}) < \infty$, then we define

$$l_k = 2 + i_k - i_{k-1}, \quad k = 1, \ldots, r(\{c_i\}) - 1, \quad l_{r(\{c_i\})} = \infty,$$

otherwise we define

$$l_k = 2 + i_k - i_{k-1}, \quad k \in \mathbb{N}.$$

We set

$$C(\{c_i\}) = \bigoplus_{k=1}^{r(\{c_i\})} A_{l_k}, \quad A_m = \begin{pmatrix} 1 & & & & 1 \\ & 0 & & & \\ & & \ddots & & \\ & & & 0 & \\ 1 & & & & 1 \end{pmatrix} \in \mathbb{C}^{m\times m},$$

where $A_\infty = \mathrm{diag}(1, 0, 0, \ldots)$. Note that $\mathrm{Sp}(C(\{c_i\})) \subset \{0, 1, 2\}$ and $1 \in \mathrm{Sp}(C(\{c_i\}))$ if and only if $r(\{c_i\}) < \infty$. Given $\{a_{i,j}\} \in \Omega'$, define the self-adjoint operator

$$A = A(\{a_{i,j}\}) = \bigoplus_{j=1}^{\infty} \left[C(\{a_{i,j}\}_{i\in\mathbb{N}}) - (1 - 2^{-(1+j)})I\right] \in \Omega_{\mathrm{SA}}.$$

If $\Xi_{2,Q}(\{a_{i,j}\}) = 1$, then $0 \notin \mathrm{Sp}(A)$. If $\Xi_{2,Q}(\{a_{i,j}\}) = 0$, there exists a null sequence in $\mathrm{Sp}(A)$ and, hence, $0 \in \mathrm{Sp}(A)$. We define the following general algorithm that uses Λ':

$$\widehat{\Gamma}_{n_2,n_1}(\{a_{i,j}\}) = 1 - \Gamma_{n_2,n_1}(A(\{a_{i,j}\})).$$

The convergence of Γ_{n_2,n_1} implies that

$$\lim_{n_2\to\infty} \lim_{n_1\to\infty} \widehat{\Gamma}_{n_2,n_1}(\{a_{i,j}\}) = \Xi_{2,Q}(\{a_{i,j}\}).$$

However, this contradicts $\{\Xi_{2,Q}, \Omega', [0,1], \Lambda'\} \notin \Delta_3^G$. □

To prove $\{\mathrm{Sp}, \Omega_{\mathrm{NL}}^{U,H}, \mathcal{M}_{\mathrm{AW}}, \Lambda_1\} \notin \Delta_3^G$, we use Proposition 10.1. Given $A \in \Omega_{\mathrm{SA}}$, we can identify it with a constant pencil $T_A \in \Omega_{\mathrm{NL}}^{U,H}$ exactly as before. Suppose for a contradiction that $\{\Gamma_{n_2,n_1}\}$ is a Δ_3^G-tower for $\{\mathrm{Sp}, \Omega_{\mathrm{NL}}^{U,H}, \mathcal{M}_{\mathrm{AW}}, \Lambda_1\}$. We now set

$$a_{n_2,n_1}(A) = \inf_{z\in\Gamma_{n_2,n_1}(A)} \min\{d_U(z, x_0), 1\},$$

where the infimum over the empty set is taken to be 1, and define

$$\tilde{\Gamma}_{n_2,n_1}(A) = 1 - a_{n_2,n_1}(A).$$

If $\Xi_0(A) = 1$, then $\lim_{n_2\to\infty} \lim_{n_1\to\infty} a_{n_2,n_1}(A) = 0$. On the other hand, if $\Xi_0(A) = 0$, then $\lim_{n_2\to\infty} \lim_{n_1\to\infty} a_{n_2,n_1}(A) = 1$. Hence, $\{\tilde{\Gamma}_{n_2,n_1}\}$ is a Δ_3^G-tower for $\{\Xi_0, \Omega_{\mathrm{SA}}, [0,1], \Lambda_1\}$, which contradicts Proposition 10.1.

Exercise 10.6

These are straightforward generalisations of Exercise 3.3 and Exercise 3.4.

Exercise 10.7

The fact stated in the exercise, together with the continuity of $z \mapsto T(z)$ in the metric $d_{\mathfrak{G}}$, show that the sets $\Delta_k(T)$ are open for $k = 1, 2, 3, 4$, and therefore so is $\Delta_5(T)$. It follows that the sets $\mathrm{Sp}_{\mathrm{ess},k}(T)$ are relatively closed in U. It is immediate that $\Delta_k(T) = \{z \in U : 0 \in \Delta_k(T(z))\}$ for $k = 1, 2, 3, 4$. Now let T be the perturbed shift in the question. For $z \neq 0$, the spectrum of the fixed operator $T(z) \in \mathscr{B}(\ell^2(\mathbb{Z}))$ is the unit circle. From the previous exercise, it follows that $\{z \in \mathbb{C} : z \neq 0\} \subset \Delta_k(T)$ for $k = 1, 2, 3, 4$. Since $T(0)$ is a rank-one perturbation of an invertible operator, it is semi-Fredholm. It follows that $\Delta_1(T) = \mathbb{C}$. We have $\mathrm{Sp}(T) = \{0\}$, and hence $\Delta_5(T) = \mathbb{C}$. Now $\Delta_1(T(0)) = \mathbb{C} \setminus \mathbb{T}$ and $\mathrm{Sp}(T(0)) = \{z \in \mathbb{C} : |z| \leq 1\}$. It follows that $\Delta_5(T(0)) = \{z \in \mathbb{C} : |z| > 1\}$.

For the example where $\mathrm{Sp}(T) \setminus \mathrm{Sp}_{\mathrm{ess},5}(T)$ need not be a discrete set, define the diagonal operator

$$T(z) = \mathrm{diag}(f(z), 1, 1, \dots) \in \mathscr{B}(\ell^2(\mathbb{N})), \qquad f(z) = \max\{|z| - 1, 0\}.$$

Clearly, $T \in \Omega_{\mathrm{NL}}^{\mathbb{C}}$. Moreover, $T(z)$ is Fredholm with index zero for all $z \in \mathbb{C}$, so $\mathrm{Sp}_{\mathrm{ess},k}(T) = \emptyset$ for $k = 1, 2, 3, 4, 5$. However, every $z \in \mathbb{C}$ with $|z| \leq 1$ is an eigenvalue of T with multiplicity one, and these points are not isolated.

Now suppose that T is holomorphic. We prove that $\mathrm{Sp_d}(B)$ is equal to $\mathrm{Sp}(T) \setminus \mathrm{Sp}_{\mathrm{ess},5}(T)$ for any suitable choice of B, thereby also showing that this definition of the discrete spectrum is independent of the choice of B. For the argument below, pick such a B. Suppose first that $z \in \mathrm{Sp}(T) \setminus \mathrm{Sp}_{\mathrm{ess},5}(T)$. Then z lies in $\mathrm{Sp}(B) \setminus \mathrm{Sp}_{\mathrm{ess},5}(B)$. Hence, it lies in a component of $\Delta_1(B)$ that intersects $U \setminus \mathrm{Sp}(B)$. Keldysh's theorem implies that $z \in \mathrm{Sp_d}(B)$. Conversely, suppose that $z_0 \in \mathrm{Sp_d}(B)$. This means that we may write

$$B(z)^{-1} = C(z) + \sum_{k=1}^{n} \frac{S_k}{(z - z_0)^k},$$

in an open neighbourhood of z_0, where S_k are finite-rank and $C(z)$ is bounded holomorphic. Suppose for a contradiction that $B(z_0) \notin \mathscr{F}_+(\mathcal{H}_3, \mathcal{H}_2)$. By the characterisation of semi-Fredholm operators in Lemma 8.1.9, there exists a sequence $\{x_m\} \subset \mathcal{H}_3$ of unit-norm vectors such that x_m converges weakly to zero as $m \to \infty$ and $\lim_{m\to\infty} \|B(z_0)x_m\| = 0$. Let $\mathcal{P}$ be the orthogonal projection onto the perpendicular of the (finite-dimensional) sum of the ranges of S_k. It follows that $\lim_{m\to\infty} \|\mathcal{P}x_m\| = 1$. Then

$$\mathcal{P}x_m = \mathcal{P}B(z)^{-1}B(z)x_m = \mathcal{P}C(z)B(z)x_m + \sum_{k=1}^{n} \mathcal{P}\frac{S_k}{(z - z_0)^k}B(z)x_m = \mathcal{P}C(z)B(z)x_m.$$

Taking $z \to z_0$ and then $m \to \infty$, we see that $\lim_{m\to\infty} \|\mathcal{P}x_m\| = 0$, the required contradiction. It follows that $B(z_0) \in \mathscr{F}_+(\mathcal{H}_3, \mathcal{H}_2)$. Using the fact stated in the exercise and the fact that z_0 is isolated, we see that $B(z_0)$ is Fredholm with $\mathrm{ind}(B(z_0)) = 0$. Moreover, $z_0 \in \mathrm{Sp}(B) \setminus \mathrm{Sp}_{\mathrm{ess},5}(B) = \mathrm{Sp}(T) \setminus \mathrm{Sp}_{\mathrm{ess},5}(T)$.

Exercise 10.8

Let $U \subset \mathbb{C}$ be a domain and $T \in \Omega_{\mathrm{NL}}^{U}$. To show the map $z \mapsto \tau_{\mathrm{inf}}(T(z))$ is continuous on U, we use the fact that

$$\tau_{\mathrm{inf}}(A_n) = \sqrt{\frac{1}{\rho_{\mathrm{ess}}(R_{A_n})} - 1}, \qquad \tau_{\mathrm{inf}}(A) = \sqrt{\frac{1}{\rho_{\mathrm{ess}}(R_A)} - 1},$$

and argue in a similar fashion as we did for the injection modulus σ_{inf}. The towers of algorithms are a simple generalisation of those for the essential spectrum in Chapter 8. Namely, we can compute τ_{inf} in two limits using Λ_2 by considering $Q_{m,n}$ and $\hat{Q}_{m,n}$, the orthogonal projections onto $\mathrm{span}\{e_{m+1}(z,T), \dots, e_n(z,T)\}$ and $\mathrm{span}\{\hat{e}_{m+1}(z,T), \dots, \hat{e}_n(z,T)\}$, respectively. We approximate $\tau_{\mathrm{inf}}(T(z))$ and $\tau_{\mathrm{inf}}([T(z)]^*)$ by $\sigma_{\mathrm{inf}}(T(z)Q_{m,n}^*)$ and $\sigma_{\mathrm{inf}}([T(z)]^*\hat{Q}_{m,n}^*)$, respectively. These approximations converge in the double limit $\lim_{m\to\infty} \lim_{n\to\infty}$. Similarly, we can adapt the lower bounds to show that

$$\Delta_3^G \not\ni \{\mathrm{Sp}_{\mathrm{ess},k}, \Omega_{\mathrm{NL}}^U, \mathcal{M}_{\mathrm{AW}}, \Lambda_1\} \in \Pi_3^A, \quad \Delta_2^G \not\ni \{\mathrm{Sp}_{\mathrm{ess},k}, \Omega_{\mathrm{NL}}^U, \mathcal{M}_{\mathrm{AW}}, \Lambda_2\} \in \Pi_2^A.$$

We now consider the wave equation with acoustic boundary conditions. Integration by parts shows that

$$[T(z)]^* u = u'' + \bar{z}^2 u, \quad \mathcal{D}([T(z)]^*) = \left\{u \in \mathcal{W}^{2,2}(\mathbb{R}_{>0}) : -u'(0) - \mathrm{i}\bar{z}u(0) = 0\right\}.$$

It is then a simple calculation to show that: if $\mathrm{Im}(z) > 0$, then $\mathrm{nul}(T(z)) = \mathrm{nul}([T(z)]^*) = 1$; if $\mathrm{Im}(z) \leq 0$, then $\mathrm{nul}(T(z)) = \mathrm{nul}([T(z)]^*) = 0$. Moreover, for z with $\mathrm{Im}(z) \neq 0$, the range of $T(z)$ is closed. It follows that

$$\{z \in \mathbb{C} : \mathrm{Im}(z) \neq 0\} \subset \Delta_k(T), \quad k = 1, 2, 3, 4.$$

Since $T(z)$ is holomorphic and none of its spectral points are isolated, the previous exercise shows that

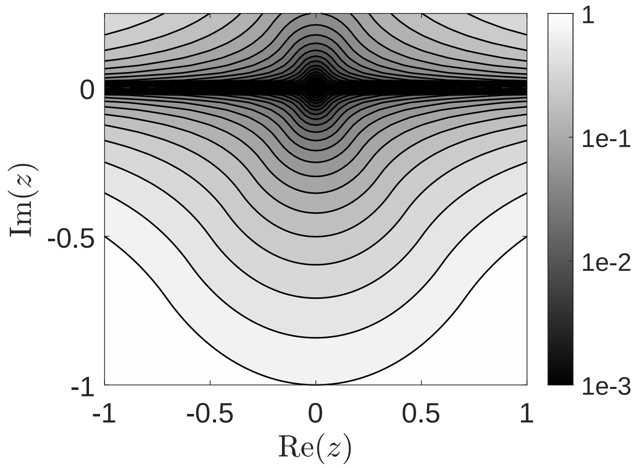
$$\mathrm{Sp}_{\mathrm{ess},5}(T) = \mathrm{Sp}(T) = \{z \in \mathbb{C} : \mathrm{Im}(z) \geq 0\}.$$

It follows that

$$\Delta_5(T) = \{z \in \mathbb{C} : \mathrm{Im}(z) < 0\}.$$

It follows that $\Delta_1(T) = \mathbb{C} \setminus \mathbb{R}$. If $z \in \mathbb{C}$ has $\mathrm{Im}(z) = 0$ and $z \neq 0$, then $\mathrm{nul}(T(z)) = \mathrm{nul}([T(z)]^*) = 0$, from which we conclude that $T(z)$ cannot have closed range. Since the sets $\Delta_k(T)$ are open, it follows that $\mathrm{Sp}_{\mathrm{ess},k}(T) = \mathbb{R}$ for $k = 1, 2, 3$.

Code for this exercise can be found in "ex10_8.m" in the repository. We use the same computational setup and basis functions as in Section 10.1.4 of the book. For this example $\sigma_{\mathrm{inf}}(T(z)\boldsymbol{Q}_{m,n}^*) = \sigma_{\mathrm{inf}}(T(z)\hat{\boldsymbol{Q}}_{m,n}^*)$, so the approximations of $\mathrm{Sp}_{\mathrm{ess},1}(T)$, $\mathrm{Sp}_{\mathrm{ess},2}(T)$, and $\mathrm{Sp}_{\mathrm{ess},3}(T)$ are the same. Here is the output for $m = 1$ and $n = 1000$, which already captures the essential spectrum:



Exercise 10.9

Let $A : U \to \mathbb{C}^{n\times n}$ be holomorphic and suppose that $\det(A(z))$ is not identically zero on U. The function $f(z) = \det(A(z))$ is holomorphic and its zeros are the eigenvalues of A. Since f is not identically zero, the eigenvalues of A are isolated. Moreover, we can use the inversion formula for a matrix to see that $A(z)^{-1}$ is finitely meromorphic. It follows that the spectrum of A is discrete. Let Υ be a piecewise smooth closed contour in U that avoids $\mathrm{Sp}(A)$, and assume that Υ has winding number 1 around $\mathrm{Sp}(A)$. The sum of the algebraic multiplicities of the eigenvalues of A is the number of zeros of the function $f(z)$ inside Υ. By the argument principle, this is equal to

$$\frac{1}{2\pi\mathrm{i}} \int_{\Upsilon} \frac{f'(z)}{f(z)}\,\mathrm{d}z.$$

Jacobi's formula shows that

$$\frac{f'(z)}{f(z)} = \mathrm{Trace}\left(\frac{\mathrm{d}A}{\mathrm{d}z}(z)A(z)^{-1}\right).$$

Since taking the trace commutes with integration, we see that

$$\frac{1}{2\pi\mathrm{i}} \int_{\Upsilon} \frac{f'(z)}{f(z)}\,\mathrm{d}z = \mathrm{Trace}\left(\frac{1}{2\pi\mathrm{i}} \int_{\Upsilon} \frac{\mathrm{d}A}{\mathrm{d}z}(z)A(z)^{-1}\,\mathrm{d}z\right),$$

and the first part of the question follows.

Now suppose that T is bounded holomorphic and define

$$L = \frac{1}{2\pi \mathrm{i}} \int_{\Upsilon} \frac{\mathrm{d}T}{\mathrm{d}z}(z) T(z)^{-1} \,\mathrm{d}z \quad \text{and} \quad S = \frac{1}{2}(L + L^*).$$

The proof that L is finite rank with $m = \mathrm{Trace}(L)$ can be found in Gohberg, Israel C., and Efim I. Sigal. *"An operator generalization of the logarithmic residue theorem and the theorem of Rouché,"* Mathematics of the USSR-Sbornik 13.4 (1971): 603 (even in the case of eigenvalues that are not semisimple; the semisimple case is an easy consequence of Keldysh's theorem). To see the equality with the trace of S, note that $\mathrm{Trace}(L^*) = \overline{\mathrm{Trace}(L)} = m$. Next, we compute the expectation of $g_j^* S g_j$ as

$$\mathbb{E}\left[g_j^* S g_j\right] = \sum_{\beta,\gamma=1}^{\infty} c_\beta c_\gamma \mathbb{E}\left[\overline{\xi_\beta} \xi_\gamma\right] \psi_\beta^* S \psi_\gamma = \sum_{\beta=1}^{\infty} c_\beta^2 \psi_\beta^* S \psi_\beta = \mathrm{Trace}(S C_j)$$

Since S has finite rank and C_j converges strongly to the identity, it follows that SC_j converges in norm to S. Since the range of these operators is inside a fixed finite-dimensional subspace, $\lim_{j\to\infty} \mathrm{Trace}(SC_j) = \mathrm{Trace}(S)$.

For the probabilistic bound, note that $\mathrm{Trace}(SC_j) = \mathrm{Trace}(\sqrt{C_j} S \sqrt{C_j})$. Since $\sqrt{C_j} S \sqrt{C_j}$ is self-adjoint and has finite rank, we may write $\sqrt{C_j} S \sqrt{C_j} = QDQ^*$ for a finite diagonal matrix $D = \mathrm{diag}(d_1, \ldots, d_M)$ and an isometry Q written in quasimatrix form as

$$Q = \begin{pmatrix} q_1 & q_2 & \cdots & q_M \end{pmatrix}.$$

We first claim that we can write

$$g_j^{(k)*} S g_j^{(k)} = \sum_{i=1}^{M} d_i |Z_{i,k}|^2,$$

where $Z_{i,k} \sim \mathcal{N}_{\mathbb{C}}(0,1)$ are independent. We have

$$g_j^{(k)*} S g_j^{(k)} = \sum_{\beta,\gamma=1}^{\infty} c_\beta c_\gamma \overline{\xi_\beta^{(k)}} \xi_\gamma^{(k)} \psi_\beta^* S \psi_\gamma = \sum_{\beta,\gamma=1}^{\infty} \overline{\xi_\beta^{(k)}} \xi_\gamma^{(k)} \psi_\beta^* \sqrt{C_j} S \sqrt{C_j} \psi_\gamma = \sum_{\beta,\gamma=1}^{\infty} \overline{\xi_\beta^{(k)}} \xi_\gamma^{(k)} \psi_\beta^* QDQ^* \psi_\gamma.$$

We then set

$$Z_{i,k} = \sum_{\beta=1}^{\infty} \xi_\beta^{(k)} [Q^* \psi_\beta]_i = \sum_{\beta=1}^{\infty} \xi_\beta^{(k)} q_i^* \psi_\beta,$$

where one can easily argue for the convergence of the series. To see independence, note that the variables are jointly Gaussian with

$$\mathbb{E}\left[Z_{i_1,k}^* Z_{i_2,k}\right] = \sum_{\beta,\gamma=1}^{\infty} \mathbb{E}\left[\overline{\xi_\beta^{(k)}} \xi_\gamma^{(k)}\right] \overline{q_{i_1}^* \psi_\beta} q_{i_2}^* \psi_\gamma = \sum_{\beta=1}^{\infty} \psi_\beta^* q_{i_1} q_{i_2}^* \psi_\beta = \mathrm{Trace}(q_{i_1} q_{i_2}^*) = \delta_{i_1 i_2}.$$

By considering the real and imaginary parts of $Z_{i,k}$, it follows that

$$X := \mathrm{tr}_K(S) - \mathrm{Trace}(\sqrt{C_j} S \sqrt{C_j}) = \frac{1}{2K} \sum_{k=1}^{2K} \sum_{i=1}^{M} d_i (Y_{i,k}^2 - 1),$$

where $Y_{i,k} \sim \mathcal{N}(0,1)$ are independent. Let

$$X_k = \frac{1}{2K} \sum_{i=1}^{M} d_i (Y_{i,k}^2 - 1), \quad k = 1, \ldots, 2K.$$

Using independence, we have

$$\log\left(\mathbb{E}\left[e^{tX_k}\right]\right) = \left(-\frac{1}{2K} \sum_{i=1}^{M} d_i t\right) + \sum_{i=1}^{M} \log\left(\mathbb{E}\left[e^{\frac{td_i}{2K} Y_{i,k}^2}\right]\right).$$

We now assume that $|td_i|/(2K) < 1/2$ and use the moment generating function of the chi-squared distribution to obtain

$$\log\left(\mathbb{E}\left[e^{\frac{td_i}{2K}Y_{i,k}^2}\right]\right) = -\frac{1}{2}\log\left(1-\frac{td_i}{K}\right).$$

We further assume that $|td_i|/(2K) < 1/4$ to obtain

$$\log\left(\mathbb{E}\left[e^{tX_k}\right]\right) = \sum_{i=1}^{M} -\frac{d_i t}{2K} - \frac{1}{2}\log\left(1-\frac{td_i}{K}\right) \leq \sum_{i=1}^{M} \frac{\left(\frac{d_i t}{2K}\right)^2}{1-2\left|\frac{d_i t}{2K}\right|} \leq \frac{t^2}{2K^2}\sum_{i=1}^{M} d_i^2 = \frac{t^2}{2K^2}\|\sqrt{C_j}S\sqrt{C_j}\|_{\mathrm{F}}^2.$$

It follows that

$$\mathbb{E}\left[e^{tX}\right] \leq \exp\left(\frac{\|\sqrt{C_j}S\sqrt{C_j}\|_{\mathrm{F}}^2}{K}t^2\right),$$

so X is sub-exponential with $\nu = \sqrt{2}\|\sqrt{C_j}S\sqrt{C_j}\|_{\mathrm{F}}/\sqrt{K}$ and $\alpha = 2\|\sqrt{C_j}S\sqrt{C_j}\|/K$. Moreover,

$$\frac{\nu^2}{\alpha} = \frac{2\|\sqrt{C_j}S\sqrt{C_j}\|_{\mathrm{F}}^2}{K}\frac{K}{2\|\sqrt{C_j}S\sqrt{C_j}\|} \geq \|\sqrt{C_j}S\sqrt{C_j}\|_{\mathrm{F}}.$$

Applying the bound in the question, we have, for every $\delta \leq \|\sqrt{C_j}S\sqrt{C_j}\|_{\mathrm{F}}$,

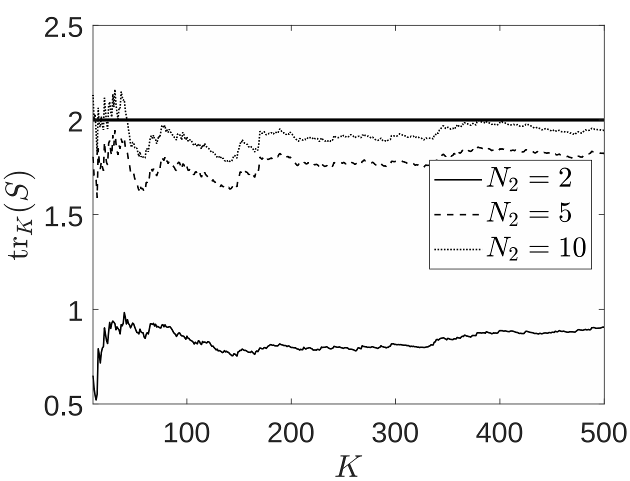
$$\mathbb{P}(X \geq \delta) \leq \exp\left(-\frac{K\delta^2}{4\|\sqrt{C_j}S\sqrt{C_j}\|_{\mathrm{F}}^2}\right).$$

Applying the same argument to $-X$ and using a union bound gives the required result.

For the Klein–Gordon example we use the same computational setup (including the contour and quadrature method) as in Section 10.3.3 of the book. For the random vectors (i.e., the choices of C_j), we use

$$\sum_{\beta=-N_2}^{N_2} \xi_\beta e_\beta, \quad \xi_n \sim \mathcal{N}_{\mathbb{C}}(0,1).$$

Code for this exercise can be found in "ex10_9.m" in the repository. The following figure shows the approximations $\mathrm{tr}_K(S)$ of the trace. The correct limiting value of 2 is visible as N_2 increases and the covariance operators converge strongly to the identity.



Exercise 10.10

We consider the function

$$F\begin{pmatrix}u\\z\end{pmatrix}=\begin{pmatrix}T(z)u\\w^*u-1\end{pmatrix}.$$

The Jacobian of this operator at (u,z) acts on (v,λ) as

$$J\begin{pmatrix}v\\\lambda\end{pmatrix}=\begin{pmatrix}T(z) & T^{(1)}(z)u\\w^* & 0\end{pmatrix}\begin{pmatrix}v\\\lambda\end{pmatrix}.$$

The update rule for Newton's method becomes

$$\begin{pmatrix}T(z_k) & T^{(1)}(z_k)u_k\\w^* & 0\end{pmatrix}\begin{pmatrix}u_{k+1}-u_k\\z_{k+1}-z_k\end{pmatrix}=-\begin{pmatrix}T(z_k)u_k\\w^*u_k-1\end{pmatrix}.$$

This rearranges to

$$T(z_k)u_{k+1}=(z_k-z_{k+1})T^{(1)}(z_k)u_k,\quad w^*u_{k+1}=1.$$

To see the equivalence with inverse iteration, let $\tilde{u}_{k+1}=[T(z_k)]^{-1}T^{(1)}(z_k)u_k=u_{k+1}/(z_k-z_{k+1})$. Then

$$w^*\tilde{u}_{k+1}=\frac{w^*u_{k+1}}{(z_k-z_{k+1})}=\frac{1}{(z_k-z_{k+1})}.$$

It follows that

$$u_{k+1}=\frac{\tilde{u}_{k+1}}{w^*\tilde{u}_{k+1}},\quad z_{k+1}=z_k-\frac{1}{w^*\tilde{u}_{k+1}}.$$

The fact that the predator-prey system is holomorphic of type (A) follows from the observation that the part of the operator that depends on z is bounded holomorphic. It is an easy check that the required conditions hold. Code for this exercise can be found in "ex10_10.m" in the repository and yields the bound stated in just a few Newton steps.

Exercise 10.11

Since $\sigma_m(\tilde{U}_0\tilde{\Sigma}_0\tilde{V}_0^*)$ can be computed to any desired accuracy, the bound in Lemma 10.3.7 implies that we can compute $\sigma_m(S(z))$ to any desired accuracy. By suitably modifying the methods of Chapter 3, one obtains a Δ_1^A algorithm for $\mathrm{Sp}_\epsilon(S)$. That is, $\Gamma_{\epsilon,\delta}(T)$ is a δ-accurate approximation of $\mathrm{Sp}_\epsilon(S)$. We compute $\Gamma_{\eta,\eta}(T)$ for successively smaller $\eta\leq\delta/4$ until its output lies in the union of a finite number of balls of radius $\delta/4$ that are separated by at least $\delta/3$. Since each connected component of the pseudospectrum must contain a point in the spectrum, this output must then lie at Hausdorff distance at most δ from the spectrum $\mathrm{Sp}(S)=\mathrm{Sp}(T)$.

11 Chapter 11

Exercise 11.1

First let $1 \le p < \infty$. As in the L^∞ case, we define

$$[\mathcal{K}g](x) = g(F(x)) \text{ for } \omega\text{-almost every } x \in \mathcal{X}.$$

This is clearly linear. To see when it is bounded, we use the Radon–Nikodym theorem to see that

$$\|\mathcal{K}g\|^p_{L^p(\mathcal{X},\omega)} = \int_{\mathcal{X}} |g(F(x))|^p \, \mathrm{d}\omega(x) = \int_{\mathcal{X}} |g(x)|^p \, \mathrm{d}(F_\#\omega)(x) = \int_{\mathcal{X}} |g(x)|^p \rho_F(x) \, \mathrm{d}\omega(x). \tag{26}$$

If $\rho_F \in L^\infty(\mathcal{X},\omega)$, it follows that $\|\mathcal{K}g\|^p_{L^p(\mathcal{X},\omega)} \le \|\rho_F\|_{L^\infty(\mathcal{X},\omega)}\|g\|^p_{L^p(\mathcal{X},\omega)} < \infty$ and the Koopman operator is bounded. Conversely, if $\|\rho_F\|_{L^\infty(\mathcal{X},\omega)} = \infty$, then since ω is σ-finite, there exists a set $S_n \subset \mathcal{X}$ with $0 < \omega(S_n) < \infty$ such that $\rho_F(x) \ge n$ for ω-almost every $x \in S_n$. By defining $k_n = \omega(S_n)$ and $g_n = k_n^{-1/p}$ on S_n and 0 otherwise for each $n \ge 1$, we have $\|g_n\|_{L^p(\mathcal{X},\omega)} = 1$ for all n but $\|\mathcal{K}g_n\|_{L^p(\mathcal{X},\omega)} \to \infty$ as $n \to \infty$.

The above also shows that $\|\mathcal{K}\| \le \|\rho_F\|^{1/p}_{L^\infty(\mathcal{X},\omega)}$ whenever $\|\rho_F\|_{L^\infty(\mathcal{X},\omega)} < \infty$. Let $\epsilon > 0$ with $\epsilon < \|\rho_F\|_{L^\infty(\mathcal{X},\omega)}$ and define $X_\epsilon = \{x \in \mathcal{X} : \rho_F(x) \ge \|\rho_F\|_{L^\infty(\mathcal{X},\omega)} - \epsilon\}$, which has $\omega(X_\epsilon) > 0$ by definition of the essential supremum. Using the fact that ω is σ-finite, we may assume that $\omega(X_\epsilon) < \infty$ without loss of generality in the following argument. Define the function g_ϵ by $g_\epsilon = 1$ on X_ϵ and 0 otherwise. Then $\|g_\epsilon\|^p_{L^p(\mathcal{X},\omega)} = \omega(X_\epsilon)$, but $\|\mathcal{K}g_\epsilon\|^p_{L^p(\mathcal{X},\omega)} \ge (\|\rho_F\|_{L^\infty} - \epsilon)\omega(X_\epsilon)$. Taking the limit as $\epsilon \to 0$ we see that $\|\mathcal{K}\| = \|\rho_F\|^{1/p}_{L^\infty(\mathcal{X},\omega)}$ exactly.

In the case $p = 2$ for every $f, g \in L^2(\mathcal{X},\omega)$,

$$\langle \mathcal{K}f, \mathcal{K}g\rangle = \int_{\mathcal{X}} f(F(x))\overline{g(F(x))} \, \mathrm{d}\omega(x) = \int_{\mathcal{X}} f(x)\overline{g(x)} \, \mathrm{d}(F_\#\omega)(x) = \int_{\mathcal{X}} f(x)\overline{g(x)}\rho_F(x) \, \mathrm{d}\omega(x).$$

The proposition now follows.

Suppose that $\rho_F(x) = 1$ for ω-almost every $x \in \mathcal{X}$. For each $1 \le p < \infty$, using (26)

$$\|\mathcal{K}g\|^p_{L^p(\mathcal{X},\omega)} = \int_{\mathcal{X}} |g(x)|^p \rho_F(x) \, \mathrm{d}\omega(x) = \int_{\mathcal{X}} |g(x)|^p \, \mathrm{d}\omega(x) = \|g\|^p_{L^p(\mathcal{X},\omega)}.$$

For the final part, let $p = \infty$ and suppose that $F_\#\omega \ll \omega$. If $\omega \ll F_\#\omega$ does not hold, then there exists a measurable set S with $\omega(S) > 0$ but $F_\#\omega(S) = 0$. It follows that

$$\|\chi_S\|_{L^\infty(\mathcal{X},\omega)} = 1, \quad \|\mathcal{K}\chi_S\|_{L^\infty(\mathcal{X},\omega)} = \|\chi_{F^{-1}(S)}\|_{L^\infty(\mathcal{X},\omega)} = 0.$$

Hence, $\mathcal{K}$ is not an $L^\infty(\mathcal{X},\omega)$-isometry. For the other case, suppose that $\omega \ll F_\#\omega$. We already know that $\|\mathcal{K}g\|_{L^\infty(\mathcal{X},\omega)} \le \|g\|_{L^\infty(\mathcal{X},\omega)}$. Let $g \in L^\infty(\mathcal{X},\omega)$ with $\|g\|_{L^\infty(\mathcal{X},\omega)} > 0$ and $\epsilon > 0$. Set $S_\epsilon = \{x : |g(x)| \ge \|g\|_{L^\infty(\mathcal{X},\omega)} - \epsilon\}$ (strictly speaking, we choose an element in the equivalence class of functions equal to g almost everywhere). By definition of the essential supremum, $\omega(S_\epsilon) > 0$ and hence $F_\#\omega(S_\epsilon) > 0$. If $x \in F^{-1}(S_\epsilon)$, then $|g(F(x))| \ge \|g\|_{L^\infty(\mathcal{X},\omega)} - \epsilon$. Since $\omega(F^{-1}(S_\epsilon)) = F_\#\omega(S_\epsilon)$, it follows that $\|\mathcal{K}g\|_{L^\infty(\mathcal{X},\omega)} \ge \|g\|_{L^\infty(\mathcal{X},\omega)} - \epsilon$. Since $\epsilon > 0$ was arbitrary, we conclude that $\|\mathcal{K}g\|_{L^\infty(\mathcal{X},\omega)} \ge \|g\|_{L^\infty(\mathcal{X},\omega)}$.

Exercise 11.2

These are some of the central results of Ridge, William C. *"Spectrum of a composition operator,"* Proceedings of the American Mathematical Society 37.1 (1973): 121-127.

Exercise 11.3

We first recall the definition of a two-sided Bernoulli shift. Let $k \in \mathbb{N}$ and consider the space $S_k = \{1, \dots, k\}$ with the discrete topology. Form the product space

$$\mathcal{X} = \prod_{n\in\mathbb{Z}} S_k = \{1, \dots, k\}^{\mathbb{Z}} = \{\{x_n\}_{n\in\mathbb{Z}} : x_n \in S_k\},$$

equipped with the product topology and product σ-algebra. Define the function $F(\{x_n\}_{n\in\mathbb{Z}}) = \{x_{n+1}\}_{n\in\mathbb{Z}}$. Given a vector $p = (p_1, \dots, p_k)$ of non-negative numbers with $\sum_{j=1}^k p_j = 1$, consider the measure $\mu = \sum_{j=1}^k p_j\delta_j$ on S_k. This induces a product measure ω and $(\mathcal{X}, \omega; F)$ is a two-sided Bernoulli shift.

We may assume that $p_j > 0$ without loss of generality (otherwise delete the corresponding symbols). Let $\{1, f_1, \dots, f_{k-1}\}$ be an orthonormal basis of $L^2(S_k, \mu)$. For each $n \in \mathbb{Z}$, define the functions $f_j^{\{n\}}(x) = f_j(x_n)$ for $j = 1, \dots, k-1$. All finite products of these functions with distinct indices, together with the function 1, yield an orthonormal basis of $L^2(\mathcal{X}, \omega)$. Call two elements of the above basis equivalent if some integer power of $\mathcal{K}_F$ carries one onto the other. The function 1 constitutes its own equivalence class; the other basis functions split into countably many equivalence classes that have the structure required for Lebesgue spectrum.

A one-sided Bernoulli shift is not invertible (in the measure-theoretic sense) and so does not have a unitary Koopman operator. Hence, it cannot have a Lebesgue spectrum.

Recall that for $\mathcal{X} = [0, 2\pi]^2_{\mathrm{per}}$ and ω the normalised Lebesgue measure on $\mathcal{X}$, the cat map is defined by

$$F(x, y) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \pmod{2\pi}.$$

Consider complex exponentials of the form $c_{m,n}e^{\mathrm{i}(mx+ny)}$ for $m, n \in \mathbb{Z}$, with $c_{m,n}$ a normalisation constant. These then form an orthonormal basis of $L^2(\mathcal{X}, \omega)$; we show moreover that after relabelling they provide a Lebesgue spectrum.

In particular, let $g_{1,0} = c_{1,0}e^{\mathrm{i}x}$ and then define $g_{1,n} = \mathcal{K}^{-n}g_{1,0}/\|\mathcal{K}^{-n}g_{1,0}\|$ for $n \in \mathbb{Z}$ (as F and hence $\mathcal{K}$ is invertible). Then let $g_{2,0}$ be the $c_{m,n}e^{\mathrm{i}(mx+ny)}$ with the smallest value of $|m| + |n|$ that is not already any of the $g_{1,j}$, and as before define $g_{2,n} = \mathcal{K}^{-n}g_{2,0}/\|\mathcal{K}^{-n}g_{2,0}\|$. Repeat this process by induction to construct $g_{m,n}$ for all $m \in \mathbb{Z}^+$ and $n \in \mathbb{Z}$. Note that no repeat elements appear in this set as F is a bijection, and every $c_{m,n}e^{\mathrm{i}(mx+ny)}$ for $m \in \mathbb{Z}$ and $n \in \mathbb{Z}$ appears at some point. Hence, we may conclude that the cat map has a Lebesgue spectrum of countably infinite multiplicity.

Exercise 11.4

The Koopman operator acts as a right shift on $\ell^2(\mathbb{Z})$. Using Fourier series, we may identify this space with $L^2([-\pi, \pi]_{\mathrm{per}})$. The Koopman operator then becomes multiplication by $e^{\mathrm{i}x}$. In particular, given $g \in L^2([-\pi, \pi]_{\mathrm{per}})$, this identification means that

$$\widehat{\xi_g}(n) = \frac{1}{2\pi}\int_{[-\pi,\pi]_{\mathrm{per}}} e^{-\mathrm{i}n\theta}\,\mathrm{d}\xi_g(\theta) = \frac{1}{2\pi}\langle g, \mathcal{K}^n g\rangle = \frac{1}{2\pi}\int_{[-\pi,\pi]_{\mathrm{per}}} |g(\theta)|^2 e^{-\mathrm{i}n\theta}\,\mathrm{d}\theta.$$

It follows that ξ_g is absolutely continuous with Radon-Nikodym derivative $\rho_g(\theta) = |g(\theta)|^2$. In particular, under our isomorphism between $\ell^2(\mathbb{Z})$ and $L^2([-\pi, \pi]_{\mathrm{per}})$, $g(k) = \sin(k)/(k\sqrt{\pi})$ becomes

$$g(\theta) = \sum_{k\in\mathbb{Z}} \frac{e^{\mathrm{i}k\theta}}{\sqrt{2\pi}} \frac{\sin(k)}{k\sqrt{\pi}},$$

which is exactly the Fourier series of $\chi_{[-1,1]}/\sqrt{2}$, completing the exercise.

Exercise 11.5

It is immediate to check that all of the Haar wavelet functions have norm 1. It is also clear that the function 1 is orthogonal to the other Haar wavelet functions. For each n, k_1, and k_2, φ_{n,k_1} and φ_{n,k_2} have disjoint support so are orthogonal. For each n_1, n_2, k_1, k_2 where $n_1 \leq n_2$ without loss of generality, the support of φ_{n_2,k_2} is contained in a region on which φ_{n_1,k_1} is constant (with value 0, 1 or -1) and so they are again orthogonal. For completeness, note that finite linear combinations of Haar wavelets can produce the indicator function of each closed interval in $[0, 1]$ with dyadic endpoints, and it is a standard result that these are dense in $L^2([0, 1])$.

Clearly $\mathcal{K}1 = 1$. Now, for $0 \leq x < 1/2$

$$\mathcal{K}\varphi_{n,k}(x) = \varphi_{n,k}(2x) = 2^{n/2}\varphi(2^{n+1}x - k) = \frac{1}{\sqrt{2}}2^{(n+1)/2}\varphi(2^{n+1}x - k) = \frac{1}{\sqrt{2}}\varphi_{n+1,k}(x).$$

Similarly, for $1/2 \le x \le 1$,

$$\begin{aligned}\mathcal{K}\varphi_{n,k}(x) = \varphi_{n,k}(2-2x) = 2^{n/2}\varphi(2^n(2-2x)-k) = \frac{1}{\sqrt{2}}2^{(n+1)/2}\varphi(1-(k+1-2^{n+1}+2^{n+1}x))\\ = -\frac{1}{\sqrt{2}}2^{(n+1)/2}\varphi(2^{n+1}x-(2^{n+1}-k-1)) = -\frac{1}{\sqrt{2}}\varphi_{n+1,2^{n+1}-(k+1)}(x),\end{aligned}$$

using that $\varphi(1-x) = -\varphi(x)$. But note that $\operatorname{supp}\varphi_{n+1,k} \subset [0,1/2]$ (as $k \in \{0,1,\dots,2^n-1\}$ and similarly $\operatorname{supp}\varphi_{n+1,2^{n+1}-(k+1)} \subset [1/2,1]$ and so we conclude that

$$\mathcal{K}\varphi_{n,k} = \frac{1}{\sqrt{2}}\varphi_{n+1,k} - \frac{1}{\sqrt{2}}\varphi_{n+1,2^{n+1}-(k+1)}.$$

This implies that $\mathcal{K}$ is not unitary as it is not surjective, as $\varphi_{1,0}$ is not in its range. However, it is an isometry (it is easily checked that $\mathcal{K}^*\mathcal{K}\varphi_{n,k} = \varphi_{n,k}$).

Suppose that $g \in \operatorname{span}\{1\}^\perp$ is a finite linear combination of the functions $\varphi_{n,k}$. The above shows that $\langle g, \mathcal{K}^n g\rangle$ vanishes for large n and hence ξ_g is a trigonometric polynomial. Such g are dense in $\{1\}^\perp$ and hence $L^2([0,1]) = \operatorname{span}\{1\} \oplus \mathcal{H}_{\mathrm{ac}}$.

Exercise 11.6

Recall that for $\mathcal{X} = [0,2\pi]_{\mathrm{per}}$ and ω a normalised Lebesgue measure, the doubling map is defined by

$$F(x) = 2x \pmod{2\pi} = \begin{cases} 2x, & \text{if } 0 \le x < \pi, \\ 2x - 2\pi, & \text{if } \pi \le x < 2\pi. \end{cases}$$

Fix a Borel measurable set S. Note that $\omega(S) = \|\chi_S\|^2$ and also $\omega(F^{-1}(S)) = \|\mathcal{K}\chi_S\|^2$. We expand χ_S in a Fourier basis as $\chi_S(x) = \sum_{n=-\infty}^{\infty} a_n e^{\mathrm{i}nx}$. Then by Parseval's identity, $\omega(S) = \sum_{n=-\infty}^{\infty} |a_n|^2$. But then $\mathcal{K}\chi_S(x) = \sum_{n=-\infty}^{\infty} a_n e^{2\mathrm{i}nx}$ for all x, and so by Parseval's identity $\omega(F^{-1}(S)) = \sum_{n=-\infty}^{\infty} |a_n|^2 = \omega(S)$, completing the proof. Alternatively, note that the doubling map is measure-preserving if and only if it is an isometry, which is shown by the same argument as the above.

Exercise 11.7

Suppose that $\mathcal{K}$ is invertible as an operator on $L^p(\mathcal{X},\omega)$ for some $1 \le p \le \infty$. Fix S with $\omega(S) < \infty$. Then $\chi_S \in L^p(\mathcal{X},\omega)$ and so as $\mathcal{K}$ is surjective, there exists $f \in L^p(\mathcal{X},\omega)$ such that $\mathcal{K}f = \chi_S$. Let $\tilde{S} = \{x \in \mathcal{X} : f(x) = 1\}$. Then $F^{-1}(\tilde{S}) = \{x \in \mathcal{X} : f(F(x)) = 1\}$, which is the same as $S = \{x \in \mathcal{X} : \chi_S(x) = 1\}$ up to a measure zero set, and so $\omega(S \setminus F^{-1}(\tilde{S})) = \omega(F^{-1}(\tilde{S}) \setminus S) = 0$. Hence, the measure-preserving system is invertible in the sense of Definition 11.1.16.

Conversely, suppose that the measure-preserving system is invertible and take $1 \le p < \infty$. As $\mathcal{K}$ is bounded on $L^p(\mathcal{X},\omega)$ (as the system is measure-preserving), it suffices to show that it is a bijection.

We begin by showing that it is injective. Let $g,h \in L^p(\mathcal{X},\omega)$ such that $\mathcal{K}g = \mathcal{K}h$. Let $S = \{x \in \mathcal{X} : g(F(x)) \neq h(F(x))\}$, so $\omega(S) = 0$. Define also $T = \{x \in \mathcal{X} : g(x) \neq h(x)\}$. Then $F^{-1}(T) = S$. But as F is measure-preserving, $\omega(S) = \omega(F^{-1}(T)) = \omega(T) = 0$, so $g = h$.

Next, we show that $\mathcal{K}$ is surjective. Functions of the form $\sum_{j=1}^n c_n\chi_{S_n}$ for sets S_n with $\omega(S_n) < \infty$ are dense in $L^p(\mathcal{X},\omega)$ for $1 \le p < \infty$. Together with the fact that $\mathcal{K}$ is continuous with closed range (it is an isometry), it follows that it suffices to show that all indicator functions of finite, measurable sets are in the range of $\mathcal{K}$. Fix a measurable set S with finite measure. Then there exists $\tilde{S}$ such that $\omega(S \setminus F^{-1}(\tilde{S})) = \omega(F^{-1}(\tilde{S}) \setminus S) = 0$. Then $\|\mathcal{K}\chi_{\tilde{S}} - \chi_S\|^p_{L^p(\mathcal{X},\omega)} = \omega(S \setminus F^{-1}(\tilde{S})) + \omega(F^{-1}(\tilde{S}) \setminus S) = 0$ and so $\mathcal{K}\chi_{\tilde{S}} = \chi_S$.

For the case of $p = \infty$, the above argument for injectivity already works. For surjectivity, we use the fact that $\omega(\mathcal{X}) < \infty$ implies that functions of the form $\sum_{j=1}^n c_n\chi_{S_n}$ for sets S_n with $\omega(S_n) < \infty$ are dense in $L^\infty(\mathcal{X},\omega)$. The argument then proceeds as before.

Exercise 11.8

For the first part, suppose first that whenever $\mathcal{K}g = g$, g is constant ω-almost everywhere. Let A be an invariant set. Then $[\mathcal{K}\chi_A](x) = \chi_A(F(x)) = \chi_A(x)$ and so χ_A is constant ω-almost everywhere. Hence $\omega(A) = 0$ or $\omega(A) = 1$. Conversely, suppose that every invariant set has measure 0 or 1. Let g be such that $\mathcal{K}g = g$. Then for $q \in \mathbb{Q}$ define

$X_q = \{x : |g(x)| \leq q\}$ (which is well-defined up to a set of measure 0). Then as $\mathcal{K}g = g$, X_q is an invariant set and so by ergodicity $\omega(X_q) = 0$ or $\omega(X_q) = 1$. Then if $r = \sup\{q \in \mathbb{Q} : \omega(X_q) = 1\}$, $|g| = r$ ω-almost everywhere. We can then argue in a similar manner for the complex argument of g.

For the second part, fix $g \in L^2(\mathcal{X}, \omega)$. Then we can write $g(x) = \sum_{n=-\infty}^{\infty} g_n e^{inx}$ for coefficients g_n. Then

$$[\mathcal{K}g](x) = \sum_{n=-\infty}^{\infty} g_n e^{in(x+a)} = \sum_{n=-\infty}^{\infty} (g_n e^{ina}) e^{inx}.$$

Then $\mathcal{K}g = g$ ω-almost everywhere if and only if $g_n e^{ina} = g_n$ for all n, which is true if and only if $e^{ina} = 1$ for all n such that $g_n \neq 0$. If $a/(2\pi)$ is irrational, $e^{ina} = 1$ for $n = 0$ only, so $\mathcal{K}g = g$ ω-almost everywhere if and only if g is constant ω-almost everywhere. Conversely, if $a/(2\pi)$ is rational, there exist $n \neq 0$ such that $na \in 2\pi\mathbb{Z}$ and so $e^{ina} = 1$, so there exists a g such that $\mathcal{K}g = g$ but g is not constant ω-almost everywhere.

Exercise 11.9

We begin by showing that a strongly mixing system is ergodic. Let A be an invariant set. Then $F^{-n}(A) \cap A = A$ for all n (up to sets of measure zero). Hence, by the strong mixing condition $\omega(A) = \omega(A)^2$ and so $\omega(A) = 0$ or $\omega(A) = 1$.

Next, suppose the system has countable Lebesgue spectrum, i.e., we have an orthonormal basis of $L^2(\mathcal{X}, \omega)$ $\{1, g_{i,j} : i \in \mathbb{N}, j \in \mathbb{Z}\}$ such that $\mathcal{K}g_{i,j} = g_{i,j-1}$. Take measurable sets S_1 and S_2. Since $\chi_{S_1}, \chi_{S_2} \in L^2(\mathcal{X}, \omega)$, we can write them as

$$\chi_{S_1} = a_0 + \sum_{i\in\mathbb{N}} \sum_{j\in\mathbb{Z}} a_{i,j} g_{i,j}, \quad \chi_{S_2} = b_0 + \sum_{i\in\mathbb{N}} \sum_{j\in\mathbb{Z}} b_{i,j} g_{i,j}.$$

Then as 1 is orthogonal to $g_{i,j}$ for all i, j, $\omega(S_1) = a_0$ and $\omega(S_2) = b_0$. Then

$$\omega(F^{-n}(S_1) \cap S_2) = \int_{\mathcal{X}} \chi_{F^{-n}(S_1)}(x) \chi_{S_2}(x) \, d\omega(x) = \int_{\mathcal{X}} [\mathcal{K}^n \chi_{S_1}](x) \chi_{S_2}(x) \, d\omega(x).$$

But then as $\mathcal{K}^n \chi_{S_1} = a_0 + \sum_{i\in\mathbb{N}} \sum_{j\in\mathbb{Z}} a_{i,j+n} g_{i,j}$, by orthonormality we obtain that

$$\omega(F^{-n}(S_1) \cap S_2) - \omega(S_1)\omega(S_2) = \sum_{i\in\mathbb{N}} \sum_{j\in\mathbb{Z}} a_{i,j+n} \overline{b_{i,j}}.$$

The quantity on the right-hand side converges to zero as $n \to \infty$ by a simple tail argument.

Exercise 11.10

The results are given in Theorems 4, 5, and 6 of Nordgren, Eric A. *"Composition operators,"* Canadian Journal of Mathematics 20 (1968): 442-449.

Exercise 11.11

We have

$$\mathbf{X} = \mathbf{U\Sigma V}^*, \quad \mathbf{K}_{\text{DMD}} = \mathbf{YX}^\dagger, \quad \tilde{\mathbf{K}}_{\text{DMD}} = \mathbf{U}^*\mathbf{YV\Sigma}^{-1}, \quad \tilde{\mathbf{K}}_{\text{DMD}}\mathbf{W} = \mathbf{W\Lambda}, \quad \mathbf{\Phi} = \mathbf{YV\Sigma}^{-1}\mathbf{W}.$$

It follows that

$$\mathbf{K}_{\text{DMD}}\mathbf{\Phi} = \mathbf{YX}^\dagger\mathbf{YV\Sigma}^{-1}\mathbf{W} = \mathbf{YV\Sigma}^{-1}\mathbf{U}^*\mathbf{YV\Sigma}^{-1}\mathbf{W} = \mathbf{YV\Sigma}^{-1}\tilde{\mathbf{K}}_{\text{DMD}}\mathbf{W} = \mathbf{YV\Sigma}^{-1}\mathbf{W\Lambda} = \mathbf{\Phi\Lambda}.$$

Now suppose that $\lambda \in \mathbb{C}$ with $\lambda \neq 0$ is an eigenvalue of $\mathbf{K}_{\text{DMD}}$ with eigenvector $\mathbf{v}$. Then

$$\mathbf{v} = \lambda^{-1}\mathbf{K}_{\text{DMD}}\mathbf{v} = \mathbf{YV\Sigma}^{-1}\mathbf{U}^*\mathbf{v}.$$

In particular, the vector $\mathbf{u} = \mathbf{U}^*\mathbf{v}$ is nonzero. Moreover,

$$\tilde{\mathbf{K}}_{\text{DMD}}\mathbf{u} = \mathbf{U}^*\mathbf{YV\Sigma}^{-1}\mathbf{U}^*\mathbf{v} = \mathbf{U}^*\mathbf{YX}^\dagger\mathbf{v} = \lambda\mathbf{U}^*\mathbf{v} = \lambda\mathbf{u}.$$

It follows that $\mathbf{u}$ is an eigenvector of $\tilde{\mathbf{K}}_{\text{DMD}}$ with eigenvalue λ.

Exercise 11.12

As suggested, we define $h_{N,M}(z,F) = \sigma_{\inf}(\mathbf{W}^{1/2}\mathbf{\Psi}_Y\mathbf{P}\mathbf{R}^{-1} - z\mathbf{Q})$ and for $\epsilon > 0$ we let $I_N^1(\epsilon) = [0, \epsilon - 1/N]$ and $I_N^2(\epsilon) = [\epsilon - 1/(2N), \infty)$. We then define $\Gamma_{N,M}(F,\epsilon)$ as follows. For $z \in \mathrm{Grid}(N) = \frac{1}{N}(\mathbb{Z} + \mathrm{i}\mathbb{Z}) \cap \{z \in \mathbb{C} : |z| \leq N\}$, let k be the largest $j \in \{1,\ldots,M\}$ such that $h_{N,j}(z,F) \in I_N^1(\epsilon) \cup I_N^2(\epsilon)$. If such a k exists with $h_{N,k}(z,F) \in I_N^1(\epsilon)$, then $z \in \Gamma_{N,M}(F,\epsilon)$ and otherwise $z \notin \Gamma_{N,M}(F,\epsilon)$.

As $j \to \infty$, $h_{N,j}(z,F)$ converges to the relevant injection moduli, the sequence $h_{N,j}(z,F)$ cannot lie in both of $I_N^1(\epsilon)$ and $I_N^2(\epsilon)$ infinitely many times; hence, $\lim_{M\to\infty} \Gamma_{N,M}(F,\epsilon) = \Gamma_N(F,\epsilon)$ exists. Then we have the inclusions:

$$\{z \in \mathrm{Grid}(N) : h_N(z,F) < \epsilon - 1/N\} \subset \Gamma_N(F,\epsilon) \subset \{z \in \mathrm{Grid}(N) : h_N(z,F) < \epsilon - 1/(2N)\} \subset \mathrm{Sp}_{\mathrm{ap},\epsilon}(\mathcal{K}).$$

In particular, the first inclusion follows as if $h_N(z,F) < \epsilon - 1/N$ then $h_{N,j}(z,F)$ lies in $I_N^1(\epsilon)$ infinitely many times (and in $I_N^2(\epsilon)$ only finitely many times), so $z \in \Gamma_N(F,\epsilon)$. The second inclusion follows as if $z \in \Gamma_N(F,\epsilon)$, then $h_{N,j}(z,F)$ can only lie in $I_N^2(\epsilon)$ finitely many times and hence we must have that $h_N(z,F) < \epsilon - 1/(2N)$. The third inclusion is immediate by definition. Then, by standard arguments from Chapter 3 it follows that $\lim_{N\to\infty} \Gamma_N(F,\epsilon) = \mathrm{Sp}_{\mathrm{ap},\epsilon}(\mathcal{K})$. Finally, convergence of the approximate point pseudospectrum to the approximate point spectrum as $\epsilon \to 0$ completes the result.

We now turn to computing the full spectrum. Since $\mathrm{Sp}(\mathcal{K}) = \mathrm{Sp}_{\mathrm{ap}}(\mathcal{K}) \cup \overline{\mathrm{Sp}_{\mathrm{ap}}(\mathcal{K}^*)}$, we want to provide a Π_4^G algorithm to compute $\overline{\mathrm{Sp}_{\mathrm{ap}}(\mathcal{K}^*)}$. Compared to $\mathrm{Sp}_{\mathrm{ap}}(\mathcal{K})$, we require an additional limit to compute the operator folding terms $\langle \mathcal{K}^* f, \mathcal{K}^* g\rangle$ (as opposed to $\langle \mathcal{K} f, \mathcal{K} g\rangle$).

Take $N_1, N_2 \in \mathbb{N}$ with $N_1 \geq N_2$ and consider G, A computed using N_1 dictionary functions and the arguments from the proof of Theorem 11.3.5. Let $R = AG^{-1}A^* \in \mathbb{C}^{N_1\times N_1}$ and $g \in \mathrm{span}\{g_1,\ldots,g_{N_2}\}$. A direct calculation shows that

$$\sum_{j,k=1}^{N_2} \overline{g_j} g_k R_{j,k} = \langle \mathcal{P}_{V_{N_1}} \mathcal{K}^* g, \mathcal{P}_{V_{N_1}} \mathcal{K}^* g\rangle.$$

It follows that

$$\sum_{j,k=1}^{N_2} \overline{g_j} g_k (R - \lambda A^* - \overline{\lambda} A + |\lambda|^2 G)_{j,k} = \langle \mathcal{P}_{V_{N_1}} (\mathcal{K}^* - \overline{\lambda}) g, \mathcal{P}_{V_{N_1}} (\mathcal{K}^* - \overline{\lambda}) g\rangle.$$

The proof from here follows much the same arguments as Theorem 11.3.5. Let

$$h_{n_2,n_1}(z,F) = \sigma_{\inf}(\mathcal{P}_{V_{N_1}}(\mathcal{K}^* - \overline{z})\mathcal{P}_{V_{N_2}}^*).$$

By Lemma 1.2.6, as we take $n_1 \to \infty$ this converges monotonically from below to

$$h_{n_2}(z,F) = \sigma_{\inf}((\mathcal{K}^* - \overline{z})\mathcal{P}_{V_{n_2}}^*),$$

which in turn converges monotonically from above to $\sigma_{\inf}(\mathcal{K}^* - \overline{z})$ as $n_2 \to \infty$. Now let $h_{n_3,n_2,n_1}(z,F)$ be a sequence of functions that we can compute such that $\lim_{n_3\to\infty} h_{n_3,n_2,n_1}(z,F) = h_{n_2,n_1}(z,F)$. For the n_3 limit we have to take care as in the previous exercise. The final limit follows from the convergence of the approximate point pseudospectrum to the approximate point spectrum.

For the case of the class $\Omega_{\mathcal{X}}^{\alpha}$ we may use the same arguments as Theorem 11.3.5 to reduce the total number of limits by 1 (essentially as we can take the n_3 and n_2 limits simultaneously). Similarly, the measure-preserving cases reduce the number of limits by 1 in each case (essentially by removing the need for the final limit). Note that if the Koopman operator is unitary and not just an isometry, then $\mathrm{Sp}_{\mathrm{ap}}(\mathcal{K}) = \overline{\mathrm{Sp}_{\mathrm{ap}}(\mathcal{K}^*)}$ and there is no difference in classification between these two components of the spectrum.

Exercise 11.13

Since $\boldsymbol{\varphi}_X = \mathbf{Q}\mathbf{\Sigma}\mathbf{Z}^*$ and $\mathbf{Z}$ is an isometry, the numerator of the residual is the norm of

$$\mathbf{W}^{1/2}(\boldsymbol{\varphi}_Y\mathbf{Z} - \lambda\boldsymbol{\varphi}_X\mathbf{Z})\mathbf{v} = \mathbf{W}^{1/2}\boldsymbol{\varphi}_Y\mathbf{Z}\mathbf{v} - \lambda\mathbf{W}^{1/2}\mathbf{Q}\mathbf{\Sigma}\mathbf{v}.$$

Since $\mathbf{\Sigma}$ is invertible,

$$\boldsymbol{\varphi}_Y\mathbf{Z} = \mathbf{Q}\mathbf{\Sigma}\mathbf{\Sigma}^{-1}\mathbf{Q}^*\boldsymbol{\varphi}_Y\boldsymbol{\varphi}_X^*\mathbf{Q}\mathbf{\Sigma}^{-1} = \mathbf{Q}\mathbf{\Sigma}\widehat{\mathbf{K}}.$$

Since $\widehat{\mathbf{K}}\mathbf{v} = \lambda\mathbf{v}$, the numerator of the proposed residual in the exercise vanishes.

Exercise 11.14

The proof for the first part is similar to that of Corollary 5.2.4 (which in turn is based on the proof of Corollary 4.2.7). However, care is needed due to the limit superior as $M \to \infty$. We may take an open subset $U \in [-\pi, \pi]_{\mathrm{per}}$ (instead of $\mathbb{T}$). Since $\mathcal{K}$ is unitary, by the spectral theorem there exists $g \in L^2(\mathcal{X}, \omega)$ with $\|g\| = 1$ such that $\xi_g(U) > 0$. We may choose a continuous function $\phi : [-\pi, \pi]_{\mathrm{per}} \to \mathbb{R}$ of compact support in U, such that $|\phi|_{C^{0,1}_{\mathrm{per}}} \leq 1$ and

$$\int_{[-\pi,\pi]_{\mathrm{per}}} \phi(\varphi)\, \mathrm{d}\xi_g(\varphi) > 0.$$

Corollary 11.5.5 shows that

$$\lim_{N\to\infty} \limsup_{M\to\infty} \left| \int_{[-\pi,\pi]_{\mathrm{per}}} \phi(\varphi)\, \mathrm{d}\xi_{\mathbf{g}}^{(N,M)}(\varphi) - \int_{[-\pi,\pi]_{\mathrm{per}}} \phi(\varphi)\, \mathrm{d}\xi_g(\varphi) \right| = 0.$$

It follows that, for sufficiently large N,

$$\liminf_{M\to\infty} \int_{[-\pi,\pi]_{\mathrm{per}}} \phi(\varphi)\, \mathrm{d}\xi_{\mathbf{g}}^{(N,M)}(\varphi) > 0.$$

Since ϕ is supported in U, this implies that $\limsup_{M\to\infty} \inf_{\lambda \in \mathrm{Sp}(\mathbf{K}_{\mathrm{mp}})} \mathrm{dist}(\lambda, U) = 0$.

The proof of the second part is similar to that of Corollary 5.2.5 (which in turn is based on the proof of Corollary 4.2.8). Since $\mathrm{Sp}(\mathcal{K})$ and E are closed, by Urysohn's lemma there exists a bounded, continuous function $\phi : [-\pi, \pi]_{\mathrm{per}} \to [0, 1]$ such that $\phi(\varphi) = 0$ if $\exp(\mathrm{i}\varphi) \in \mathrm{Sp}(\mathcal{K})$ and $\phi(\varphi) = 1$ if $\varphi \in E$. Then we have that

$$\begin{aligned}
\|\mathbf{\Psi}\mathcal{E}_{N,M}(E)\mathbf{g}_{N,M}\|^2 &= \left\| \mathbf{\Psi} \int_E \phi(\varphi)\, \mathrm{d}\mathcal{E}_{N,M}(\varphi)\mathbf{g}_{N,M} \right\|^2 \\
&= \left\| \mathbf{\Psi} \int_{[-\pi,\pi]_{\mathrm{per}}} \phi(\varphi)\, \mathrm{d}\mathcal{E}_{N,M}(\varphi)\mathbf{g}_{N,M} \right\|^2 - \left\| \mathbf{\Psi} \int_{[-\pi,\pi]_{\mathrm{per}} \setminus E} \phi(\varphi)\, \mathrm{d}\mathcal{E}_{N,M}(\varphi)\mathbf{g}_{N,M} \right\|^2 \\
&\leq \left\| \mathbf{\Psi} \int_{[-\pi,\pi]_{\mathrm{per}}} \phi(\varphi)\, \mathrm{d}\mathcal{E}_{N,M}(\varphi)\mathbf{g}_{N,M} \right\|^2 = \left\| \mathbf{\Psi} \int_{[-\pi,\pi]_{\mathrm{per}}} \phi(\varphi)\, \mathrm{d}\mathcal{E}_{N,M}(\varphi)\mathbf{g}_{N,M} - \int_{[-\pi,\pi]_{\mathrm{per}}} \phi(\varphi)\, \mathrm{d}\mathcal{E}(\varphi)\mathbf{g} \right\|^2,
\end{aligned}$$

where the second equality holds by the orthogonality of spectral projections and the first and last by the definition of ϕ. Corollary 11.5.5 implies that $\lim_{N\to\infty} \limsup_{M\to\infty} \|\mathbf{\Psi}\mathcal{E}_{N,M}(E)\mathbf{g}_{N,M}\| = 0$.

Exercise 11.15

The first part follows by combining Proposition 11.6.6 and the functional calculus from Chapter 5, as well as the results for computing point spectra from Chapter 6. In particular, if g is a finite linear combination of the functions g_j, then we may compute $W_{n_2,n_1}(g)$ such that

$$\lim_{n_1\to\infty} W_{n_2,n_1}(g) = W_{n_2}(g), \quad \lim_{n_2\to\infty} W_{n_2}(g) = \|\mathcal{P}_{\mathrm{pp}} g\|^2 = \mu_g^{(\mathrm{pp})}(\mathbb{T}),$$

where $\mu_g^{(\mathrm{pp})}$ is the pure point part of the spectral measure of $\mathcal{K}_F$ with respect to g and the convergence as $n_2 \to \infty$ is from below. Now let $\{\tilde{g}_k\}_{k=1}^{\infty}$ be a set of such g that densely fill $\mathrm{span}\{1\}^{\perp}$ (we can achieve this by letting g_1 be a constant function and using $\{g_2, g_3, \ldots\}$ to form $\{\tilde{g}_k\}_{k=1}^{\infty}$). We then set

$$a_{n_2,n_1}(F) = \max_{1\leq k\leq n_2} W_{n_2,n_1}(\tilde{g}_k).$$

Define the two separated intervals $I_1 = [0, 1/4]$ and $I_2 = [1/2, \infty)$. As $n_1 \to \infty$, $a_{n_2,n_1}(F)$ converges to $a_{n_2}(F) = \max_{1\leq k\leq n_2} W_{n_2}(\tilde{g}_k)$ and hence cannot visit both I_1 and I_2 infinitely often. For a given n_1, we set $\Gamma_{n_2,n_1}(F) = 1$ if the largest $l = 1, \ldots, n_1$ with $a_{n_2,l} \in I_1 \cup I_2$ has $a_{n_2,l} \in I_1$. If no such l exists, or $a_{n_2,l} \in I_2$, we set $\Gamma_{n_2,n_1}(F) = 0$. If $\Xi_p^{\mathrm{dec}}(F) = 1$, then $\mu_{\tilde{g}_k}^{(\mathrm{pp})}(\mathbb{T}) = 0$ for all k and hence $a_{n_2}(F) = 0$ for all n_2. It follows that $\lim_{n_1\to\infty} \Gamma_{n_2,n_1}(F) = 1$. If $\Xi_p^{\mathrm{dec}}(F) = 0$, then there exists $\tilde{g}_k$ with $\mu_{\tilde{g}_k}^{(\mathrm{pp})}(\mathbb{T}) > 1/2$ and hence $a_{n_2}(F) > 1/2$ for sufficiently large n_2. It follows that $\lim_{n_1\to\infty} \Gamma_{n_2,n_1}(F) = 0$ for sufficiently large n_2. Moreover, since $a_{n_2,n_1}(F)$ are increasing in n_2, $\Gamma_{n_2}(F)$ is decreasing in n_2. It follows that $\{\Xi_p^{\mathrm{dec}}, \Omega', \mathcal{M}_{\mathrm{dec}}, \Lambda_{\mathcal{X}}\}^{\Delta_1} \in \Pi_2^G$.